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On the Inequitable Nature of Core Allocations*

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1. INTRODUCTION

In their paper [2] concerned with the convergence of the core to the set of competitive equilibria, Debreu and Scarf considered an economy in which the individuals could be divided into types according to their tastes and endowments. It was further assumed that there were the same number of individuals in each type. This assumption led to the conclusion [2, Theorem 2] that at each point in the core, all members of the same type receive the same allocation. If this is true for an economy, we shall say that it possesses the *equal treatment property*. Such a result was necessary in order to use their methods in proving the convergence theorem. But it is an interesting question to try to classify those economies in which it is true when the symmetry between types is relaxed.

Thus, we first prove a rather simple generalization of the Debreu–Scarf equal treatment theorem which extends this property to a wider class of economies. The main result, however, is a converse theorem which holds in the absence of our generalized symmetry conditions.

It should be noted that we are considering economies of fixed, finite size and are not concerned herein with the question of whether any inequality in treatment disappears as the market becomes larger. Our aim is to ascertain whether the core is an equitable solution concept in the above sense. It should be noted that the set of competitive equilibria always has the equal treatment property as do certain game theoretic solution concepts such as the Shapley value and the Nash solution.

The theorem we obtain also has a by-product of interest. In several ways, it is possible that the importance of the by-product exceed the importance of our (supposedly) main result. It turns out that a sufficient condition (given convexity assumptions, etc.) for the unequal treatment theorem to hold is that the points in the core cannot be attained through

* This is a revised version of Chapter 3 in [4].

the independent actions of two disjoint subeconomies. This property (defined formally in Section 4) is called *strong-superadditivity*. In a recent paper by Dreze, Gabszewicz, Schmeidler and Vind [3], the possibility of blocking all noncompetitive (i.e., nonsustainable by a price system) allocations is studied. The authors introduce the concept of fictitious traders who allow all noncompetitive allocations to be blocked. But at core allocations, the ficititious traders are collectively self-sufficient. Thus strong-superadditivity fails to hold throughout the core. Our theorem implies that this result is, in a sense to be described, very rare. Strong-superadditivity almost always holds (for at least some points) in the core.

The strong-superadditivity condition has some constructive value as well. In [4, Chaps. 4 and 5] an adjustment process was defined as a natural consequence of the behavioral postulates on the agents in a system characterized by pure barter (i.e., a bargaining model for which blocking by coalitions is the behavioral postulate and the core is the resulting solution). A proof of the stability of this system was given. A crucial assumption in this proof was the strong-superadditivity condition. If strong-superadditivity holds at a point in the core, then the core has the maximal possible dimension (in the utility space). Without this property, it is not possible to show that the coalition of all traders will eventually become the only blocking coalition. In the proof of the stability result used, this was essential.

2. The Model

The economy is divided into T types such that two individuals in the same type have the same preferences and endowments. Let there be $r_t > 0$ individuals in type t, and let $n = \sum r_t$ be the number of participants in the economy. The set of all participants is denoted N. Nonnull subsets of N are coalitions denoted S. S may be represented by a vector $s = (s_1, ..., s_T)$ where there are s_t individuals of type t in the coalition S. Such a vector is called the profile of S.

We consider pure trade economies with k commodities and assume (though this can be easily generalized) that all consumption sets are the nonnegative orthant of the commodity space R_+^k. Endowments are nonzero vectors in R_+^k, denoted ω_t as the endowment common to all members of type t. $\omega = (\omega_1, ..., \omega_T)$ is the endowment profile of the economy.

With respect to preferences, we assume that each type has strictly convex, strictly monotone, continuous preferences and the the resulting demand function, $F_t(p, p \cdot \omega_t)$ is continuously differentiable in both prices and income. We write

$$D_t(p, \omega_t) = F_t(p, p \cdot \omega_t) - \omega_t$$

as the excess demand function for type t. If S is a coalition with profile s, then

$$D_S(p, \omega) = \sum_{t=1}^{T} D_t(p, \omega_t)\, s_t$$

is its excess demand function. We denote the set of price equilibria for the economy consisting of the members of coalition S as $E_S(\omega) = \{p \in R^k \mid D_S(p, \omega) = 0\}$. Under the assumptions we have made, it is well known that $E_S(\omega)$ is nonempty for each S and that $p \in E_S(\omega)$ implies $p > 0$ because of the monotonicity of preferences.[1]

3. A Simple Extension of the Debreu-Scarf Theorem

The following is a generalization of Theorem 2 in Debreu and Scarf [2]. We shall discuss the importance of this result in Appendix A. Suffice it to say at present that this is the most general formulation in which the conclusion of the equal treatment theorem holds.

THEOREM. *If the greatest common divisor* (g.c.d.) *of* $\{r_t\}$ *is not one, then the equal treatment property holds.*

Proof. Let x be an allocation in the core and let i and j be two individuals of the same type in an economy in which g.c.d. $\{r_t\} = d > 1$. We can relabel the participants so that each type t is subdivided into r_t/d new types with d individuals in each. Further, since $d > 1$, it can be arranged through a simple permutation that i and j are placed in the same type in the subdivided economy. Now the core is independent of how the participants are labeled, and the subdivided economy satisfies the hypotheses of the Debreu–Scarf equal treatment theorem .Thus i and j are treated identically throughout the core.

We now turn to the primary topic of the paper which is to obtain a converse to the above proposition.

[1] We adopt the following convention for inequalities between vectors: $x > y$ implies $x_i > y_i$ for all i. $x \geqslant y$ implies $x_i \geqslant y_i$ for all i and $x_i > y_i$ for at least one i. $x \geqq y$ implies $x_i \geqslant y_i$ for all i.

4. The Unequal Treatment Theorem

Our main theorem can be stated as follows:

THEOREM. *If* g.c.d. $\{r_t\} = 1$, *then the set of all endowment profiles for which the corresponding economies possess the equal treatment property is contained in a closed set of Lebesgue measure zero in* R_+^{Tk}.

Thus the theorem we seek is not a complete converse to the above. To see why we might have to rule out a set of measure zero, consider an endowment profile ω which is a Pareto optimum. Thus the core is $\{\omega\}$ and the equal treatment property holds trivially, irrespective of g.c.d. $\{r_t\}$.

Corresponding to each preference pattern $\circledge_t$, we choose a continuous utility representation u_t arbitrarily:

$$u_t : R_+^k \to R.$$

Having made this selection, the u_t remain fixed. Thus, to each allocation of commodoties x there corresponds a point in the utility space R^n. We let $V(S)$ be the set of utility vectors attainable by coalition S within the limits of its own resources. $V(S) \subset R^S$, the coordinate subspace of R^n indexed by the members of S. Also, define $\bar{V}(S) = \{z \in V(S) \mid z' \in R^S, z' \geqslant z \textit{ implies } z' \notin V(S)\}$. $\bar{V}(S)$ is the set of all Pareto optima, relative to the subeconomy consisting of the members of S.

Under our assumptions, $V(S)$ has the following properties:

PROPERTY 1. *$V(S)$ is closed and bounded.*

Proof. This follows from the continuity of the u_t and the compactness of the set of allocations attainable by S.

PROPERTY 2. *If* $z \in \bar{V}(S)$, $z_i > u_i(0)$ *for all* $i \in S$, *and* $\xi \geqslant 0$ *where* $\xi \in R^S$, *then for some* $\delta > 0$ *sufficiently small,* $z - \delta\xi \in \operatorname{int} V(S)$.

Proof. This follows directly from the assumption of strict monotonicity of preferences.

This property is closely related to the assumption of "openness" used in the proof of the convergence of the mechanism studied in [6]. Because of the above properties, the core $\mathscr{C}$ is defined as

$$\mathscr{C} = \{z \in V(N) \mid z\,|_S \notin \operatorname{int} V(S) \text{ for all } S\},$$

where $z\,|_S$ is the projection of z into R^S. We shall say that the condition of *strong-superadditivity* is satisfied *at a point* $u \in \mathscr{C}$ *if*

$$u\,|_S \notin V(S) \qquad \text{for all} \quad S \neq N.$$

The condition of strong-superadditivity means that not only is u unblocked, but even if the power of some coalition (except N) were to increase slightly, u would remain unblocked. The proof of the main theorem is based on a line of argument proceeding in the following three stages.

(1) If the existence of a point in the core at which the condition of strong-superadditivity holds could be demonstrated, it can be shown that the economy fails to possess the equal treatment property.

(2) If for all $S \neq N$, $E_S(\omega) \cap E_{S^c}(\omega) = \varnothing$, and u^* is a utility allocation associated with a competitive equilibrium then the condition of strong-superadditivity can be shown to hold at u^*.

(3) The set of all $\omega \in R_+^{Tk}$ for which there exists $S \neq N$ such that $E_S(\omega) \cap E_{S^c}(\omega) \neq \varnothing$ is a closed set of Lebesgue measure zero in R_+^{Tk} if the profile vector of the economy, $r = (r_1, ..., r_T)$ satisfies, g.c.d. $\{r_t\} = 1$.

Clearly, (1), (2) and (3) imply the theorem stated.

LEMMA 1. *If the condition of strong-superadditivity is satisfied at* $u^* \in \mathscr{C}$, *then the economy fails to possess the equal treatment property.*

Proof. If there exists a type containing two individuals, i and j and $u_i^* \neq u_j^*$, this suffices. Assume that $u_i^* = u_j^*$ for all pairs (i, j) of individuals in the same type. By the condition of strong-superadditivity and the fact that the $V(S)$ are closed, there exists $\epsilon > 0$ such that $z \in N_\epsilon(u^*)$ implies $z|_S \notin V(S)$ for all $S \neq N$. Let $\delta_i = (0, ..., 1, ..., 0)$ with the 1 in the ith place. Thus $y = u^* - (\epsilon/2)\,\delta_i \in N_\epsilon(u^*)$ and, by property (2), $y \in \operatorname{int} V(N)$ for ϵ sufficiently small. Now $y_i \neq y_j$ by construction. Further, $y|_S \notin V(S)$ for all $S \neq N$ since $y \in N_\epsilon(u^*)$. Let $\bar{\alpha} = \sup\{\alpha \mid y + \alpha\delta_j \in V(N)\}$. $\bar{\alpha}$ is finite since $V(N)$ is bounded. Since $V(N)$ is closed $\bar{u} = y + \bar{\alpha}\delta_j \in \overline{V}(N)$. By the above, $\bar{u} \in \mathscr{C}$ and $\bar{u}_i < \bar{u}_j$ which suffices to prove the lemma.

LEMMA 2. *If* $E_S(\omega) \cap E_{S^c}(\omega) = \varnothing$ *for all* $S \neq N$, *and if* u^* *is the utility allocation associated with a competitive equilibrium for the economy, then the condition of strong-superadditivity is satisfied at* u^*.

Proof. Let p be the competitive price vector and x the competitive allocation associated with u^*. $p \notin E_S(\omega)$ for all $S \neq N$ since if it were we would have $p \in E_{S^c}(\omega)$ by the additivity of the excess demand function, contradicting the hypothesis of the lemma. It suffices to show that $u^*|_S \notin V(S)$. Let $G_i(x_i) = \{y_i \mid y_i \succcurlyeq_i x_i\}$. Thus we must show that $\sum_S \omega_i \notin \sum_S G_i(x_i)$. p is positive by strict monotonicity. Therefore, by strict convexity of preferences, $y_i \in G_i(x_i)$ implies $p \cdot y_i \geqslant p \cdot x_i$, with equality holding only for $y_i = x_i$. Thus, for all $\{y_i\}$ such that $y_i \in G_i(x_i)$,

$$p \cdot \sum_S y_i \geqslant p \cdot \sum_S x_i = p \cdot \left(D_S(\omega) + \sum_S \omega_i\right).$$

$p \cdot D_S(\omega) = 0$ by Walras' law. If $\sum_S y_i = \sum_S \omega_i$ equality holds above. But this equality can hold only if $y_i = x_i$ for all $i \in S$, which contradicts $D_S(\omega) \neq 0$. Thus, $u^* |_S \notin V(S)$.

It, therefore, suffices to prove that the set of all endowment profiles ω for which there exist a coalition S and a price vector p such that p is an equilibrium for both S and S^C is contained in a closed subset of measure zero in R_+^{Tk}.

Let $d \in R^{Tk}$ be an excess demand profile. That is $d = (d_1, ..., d_T)$ and $d_t \in R^k$ is the excess demand of each of the individuals of type t. Define

$$f: R_+^k \times R_+^{Tk} \to R^{Tk}$$

by $f(p, \omega) = d$. Then f gives the excess demand profiles corresponding to given prices and endowment profiles.

Let

$$\Delta_S = \left\{d \in R^{Tk} \;\middle|\; \sum_t^T s_t d_t = \sum_t^T (r_t - s_t)\, d_t = 0\right\}.$$

Δ_S is the set of all excess demand profiles that are simultaneously market clearing for a coalition S and its complement.

LEMMA 3. *If* g.c.d. $\{r_t\} = 1$, $s = (s_1, ..., s_T)$ *and* $r = (r_1, ..., r_T)$ *with* $0 \leqslant s \leqslant r$, *then there exists no* $\alpha \geqslant 0$ *such that* $s_t = \alpha(r_t - s_t)$ *for all* t.

Proof. Suppose it is not so, then $(1 + \alpha)\, s_t = \alpha r_t$ or $s_t = \alpha(1/1 + \alpha)\, r_t$. Let $\beta = \alpha/(1 + \alpha)$, so that $0 < \beta < 1$. If $s_{t'} = 0$ for some t', then $r_{t'} = 0$, contradicting our assumption that $r > 0$. Similarly, $r_t - s_t > 0$ for all t or else $s = r$, which contradicts $s \leqslant r$. Since r_t and s_t are integers, α is rational. Hence β is rational, say $\beta = p/q$ reduced to the lowest terms. Thus $pr_t = qs_t$ for all t and $q > 1$ since $0 < \beta < 1$. But then q divides r_t for all t since p and q have no common factor, contradicting g.c.d. $\{r_t\} = 1$.

LEMMA 4. *Δ_S has dimension $Tk - 2k$.*

Proof. Let $d = (d_1, ..., d_T) = (d_{11}, ..., d_{1k}, ..., d_{T1}, ..., d_{Tk}) \in R^{Tk}$, where

d_{ti} is excess demand for commodity i by type t. Δ_S is the linear subspace of R^{Tk} satisfying

$$s_1 d_{1i} + \cdots + s_T d_{Ti} = 0, \qquad i = 1,\ldots, k,$$

and

$$(r_1 - s_1)\, d_{1i} + \cdots + (r_T - s_T)\, d_{Ti} = 0 \qquad i = 1,\ldots, k.$$

By the lemma above, these equations are independent and therefore Δ_S is a linear subspace of dimension $Tk - 2k$.

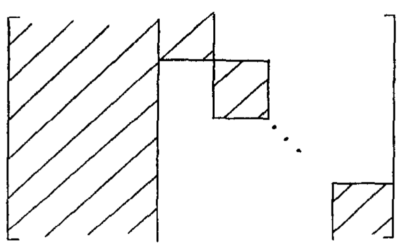
If there exists (p, ω) such that $f(p, \omega) \in \Delta_S$, then this ω violates the condition of strong-superadditivity at the point in the core which is the equilibrium associated with the price vector p. We are thus concerned with finding the set of all $\omega \in R_+^{Tk}$ for which there exists a p and S such that $f(p, \omega) \in \Delta_S$. We shall show that the set of all such ω is closed and null in R_+^{Tk} and, therefore, that the property of strong-superadditivity holds at all of the equilibrium utility allocations for economies with endowments outside this set.

Consider the Jacobian of $f(p, \omega)$, defined for all $p \geqslant 0$ and $\omega \geqslant 0$ by

$$\mathcal{J}(p, \omega) = \left[\frac{\partial f^{ti}}{\partial p_j} \;\middle|\; \frac{\partial f^{ti}}{\partial \omega_{t'j}}\right],$$

where $\partial f^{ti}/\partial p_j$ is the partial derivative of the excess demand function for commodity i by individuals of type t with respect to the price of commodity j. $\partial f^{ti}/\partial \omega_{t'j}$ is the derivative with respect to the holdings of commodity j in the endowment of individuals of type t'. Thus the left-hand submatrix is $Tk \times k$ and the right-hand submatrix is $Tk \times Tk$. (That is, row indices denote components of the excess demand profile and column indices denote the various arguments of the function f.)

As a consequence of the absence of consumption externalities, $\mathcal{J}$ has the following structure:

$\mathcal{J}$ = 

Areas outside the shaded region have all zero entries. Nonzero parts of the right-hand submatrix occur on $k \times k$ blocks along the diagonal, reflecting the fact that endowment variations affect only the excess demand vector of the individuals whose endowment has changed.

Consider the matrices made up of the k rows of $\mathscr{J}$ corresponding to the excess demand vector of type t.

$$\mathscr{J}^t(p, \omega_t) = [\mathscr{J}_1{}^t \mid \mathscr{J}_2{}^t],$$

where both submatrices are $k \times k$, $\mathscr{J}_1{}^t$ being the appropriate rows of the left-hand submatrix of $\mathscr{J}$ and $\mathscr{J}_2{}^t$ is the corresponding block on the diagonal of the right-hand submatrix of $\mathscr{J}$.

LEMMA 5. *$\mathscr{J}_2{}^t$ has rank $k - 1$ at all (p, ω), $p > 0$, $\omega_t \geqslant 0$.*

Proof. We let $p > 0$ be fixed and consider $\omega_t > 0$ first. $\mathscr{J}_2{}^t$ cannot have rank k because $p \cdot (\mathscr{J}_2{}^t\, d\omega_t) = 0$ by the budget constraint. Therefore $\mathscr{J}_2{}^t\, d\omega_t$ lies in the orthogonal complement of the one-dimensional subspace spanned by p. This orthogonal complement has dimension $k - 1$ which is, therefore, the maximal rank of $\mathscr{J}_2{}^t$. Suppose that $\mathscr{J}_2{}^t$ had rank less than $k - 1$ at some $\bar{\omega}_t > 0$. Select a small vector dz in the orthogonal complement of the space spanned by the columns of $\mathscr{J}_2{}^t$ but such that $p \cdot dz = 0$. We shall show that there exists $d\omega_t$ such that $dz = \mathscr{J}_2{}^t\, d\omega_t$ lies outside the subspace spanned by the columns of $\mathscr{J}_2{}^t$ evaluated at $(p, \bar{\omega}_t)$. This contradicts the fact that this subspace has dimension less than $k - 1$.

Consider $G(p, \bar{\omega}_t) = \{\omega_t \mid p \cdot \omega_t = p \cdot \bar{\omega}_t\}$. By definition of the demand function F_t, we have

$$F_t(p, p \cdot \omega_t) = F_t(p, p \cdot \bar{\omega}_t)$$

for all $\omega_t \in G(p, \omega_t)$. Thus $D_t(p, \omega_t) = D_t(p, \bar{\omega}_t) + (\bar{\omega}_t - \omega_t)$. Let $d\omega_t = \omega_t - \bar{\omega}_t = -dz$. We can reduce dz by a scale factor, if necessary, so that $\omega_t \in R_+^k$, because $\bar{\omega}_t > 0$ by assumption. Since $p \cdot dz = 0$, we have $\omega_t \in G(p, \bar{\omega}_t)$. Thus,

$$(D_t(p, \bar{\omega}_t + d\omega_t) - D_t(p, \bar{\omega}_t)) \cdot p = 0$$

and hence, $\mathscr{J}_2{}^t\, d\omega_t = dz$.

We now extend this result to all $\omega_t \in R_+^k$. Extend $D_t(p, \cdot)$ to all $\omega_t \in R^k$ such that $p \cdot \omega_t > 0$ by $D_t(p, \omega_t) = F_t(p, p \cdot \omega_t) - \omega_t$. The derivatives of D_t exist everywhere on the boundary of R_+^k since $\omega_t \geqslant 0$ and $p > 0$. Thus the argument used above can be extended directly to all nonzero, nonnegative ω_t.

LEMMA 6. *The columns of $\mathscr{J}_1{}^t$ span a subspace not contained in that spanned by the columns of $\mathscr{J}_2{}^t$.*

Proof. By the above lemma $p \cdot \mathscr{J}_2^t\, d\omega = 0$ for all $d\omega$. Thus it suffices to show that for all neighborhoods U of p, there exists $p' \in U$ such that $p \cdot \mathscr{J}_1^t \cdot (p' - p) \neq 0$. Consider a change Δp in prices such that the excess demand vector bought at p is still obtainable. This is clearly possible for $f^t \neq 0$ by decreasing a price for which f^t had a positive component. If $f^t = 0$, then any price change will do. (Note that by Walras' law, $f^{ti} < 0$ for some i implies $f^{tj} > 0$ for some j.) Let $f^{t'}$ be the excess demand vector corresponding to $p' = p + \Delta p$. Then $p \cdot f^t = 0$ and $p' \cdot f^{t'} = 0$. Consider $p \cdot \Delta f^t = p \cdot f^{t'} - p \cdot f^t = p \cdot f^{t'} : p \cdot f^{t'} > 0$ by the weak axiom of revealed preference, which is known to hold since we are dealing with consumers whose preferences are representable by numerical utility functions. Thus the Jacobian of the excess demand function for type t does not have all its columns lying in the orthogonal complement of p.

Thus the rank of the Jacobian of f is Tk, since the submatrices corresponding to various types $\mathscr{J}^t$ can be seen to be independent from the structure of $\mathscr{J}$ (see p. 7). Each $\mathscr{J}^t$ has rank k because $\mathscr{J}_2^t$ has rank $k - 1$ and $\mathscr{J}_1^t$ increases this, but there are only k rows in $\mathscr{J}^t$.

We now select a basis for R^{Tk} as follows: Let the first $Tk - 2k$ vectors span Δ_S. This is possible by Lemma 4. Then choose the last $2k$ so that they extend this to a basis for R^{Tk}. Let these vectors be denoted

$$a_S^1,\dots, a_S^{Tk-2k}, a_S^{Tk-2k+1},\dots, a_S^{Tk}$$

when expressed in the usual orthogonal basis so that $a_S^i \in R^{Tk}$ for all i and the a_S^i are linearly independent.

Let A_S be the $Tk \times Tk$ matrix with columns a_S^i. Let

$$g^S\colon R_+^k \times R_+^{Tk} \to R^{Tk}$$

be the function given by

$$g^S(p, \omega) = A_S \cdot f(p, \omega),$$

where $f(p, \omega)$ is the excess demand profile at (p, ω) expressed as a column vector.

By construction of A_S, if $f \in \Delta_S$, then $g_i^S = 0$ for

$$i = Tk - 2k + 1,\dots, Tk$$

and, conversely. Consider the system of $2k$ implicit functions

$$g_i^S(p, \omega) = 0 \qquad i = Tk - 2k + 1,\dots, Tk.$$

The Jacobian of g has rank Tk since A_S is nonsingular and the Jacobian of f has rank Tk.

Therefore, the rank of the submatrix of the Jacobian of g consisting of the last $2k$ rows is $2k$, since all the rows of the Jacobian of g are linearly independent. We now apply the implicit parameterization theorem (see Appendix B) to the functions g_i^S, $i = Tk - 2k + 1,..., Tk$ and the set Δ_S. This yields that the set of (p, ω) such that $f(p, \omega) \in \Delta_S$ is a differentiable manifold of dimension $Tk - k$. But by construction, Δ_S is the set of d such that p is an equilibrium for both S and S^c. Thus,

$$f^{-1}(\Delta_S)|_\omega = \{\omega \in R_+^{TK} \mid (p, \omega) \in \Delta_S \text{ for some } p\}$$

is the set of all endowment patterns that could possibly lead to price equilibria for both S and S^C simultaneously. It is clear that the projection of $f^{-1}(\Delta_S)$ onto the subspace $\{(p, \omega) \mid p = 0\}$ will not increase its dimension. Thus for each $S \subset N, f^{-1}(\Delta_S)$ is a subspace of dimension $Tk - k$. Thus

$$\bigcup_{S \subset N} f^{-1}(\Delta_S)|_\omega$$

which is the set of all such potentially pathological endowment profiles is a manifold of this dimension and hence of Lebesgue measure zero in R_+^{Tk}.

Appendix A

We shall examine certain implications of Theorem 1, the generalization of the Debreu–Scarf theorem to the case g.c.d. $\{r_t\} \neq 1$. Assume that there are only two types. Then in a large economy with an even number of traders, say $2n$, only in the case in which there are n of each type does the Debreu–Scarf theorem apply. Clearly as n gets large, the proportion of all such economies to the total number of economies with two types and $2n$ individual approaches zero. However, the hypothesis of our Theorem 1 holds in a nonzero proportion of all economies with two types in the following sense:[1]

Let $\phi(n)$ be the number of integers less than or equal to n and relatively prime to n (i.e., not sharing a common factor with n). Let $\Phi(n) = \sum_i^n \phi(i)$. Then $\Phi(n)$ is the number of relatively prime pairs in the triangular lattice of integers $\{(m, n) \mid 0 < m \leqslant n\}$. There are $n(n + 1)/2$ points in such a set. Then by a theorem in Hardy and Wright (Theorem 332)

$$\lim_n \frac{\Phi(n)}{n(n + 1)/2} = \frac{6}{\pi^2}.$$

[1] The following is taken from Hardy and Wright [5].

This limit approximates the probability that a large market with two types will satisfy g.c.d. $\{r_1, r_2\} = 1$. Thus, since $6/\pi^2 < 1$, there is a nonzero probability that, even in a very large market, the hypothesis of our generalization will apply and that the equal treatment result will hold. Our generalization thereby extends the equal treatment property to a considerably wider class of economies.

If there are more than two types, the above can be generalized as follows[2]

Let $P(k) = \lim_{N\to\infty} (\text{Prob}(\text{g.c.d. } (x_1, ..., x_k) = 1), 0 < x_i \leqslant N \text{ for all } i)$, where $P(k)$ is the "probability that a k-tuple of integers do not share a common factor." Then $P(k) = 1/\zeta(k)$ where $\zeta(k) = \sum_{m=1}^{\infty} 1/m^k$ is the Riemann zeta-function. Thus $P(k) < 1$ for all k and $\lim_k P(k) = 1$.

Thus the probability, in some sense, that the generalization of the Debreu–Scarf equal treatment theorem is inapplicable is very high when there are a large number of types. This means that the main theorem of this paper becomes increasingly relevant.

Appendix B

The implicit parameterization theorem, a generalized form of the implicit function theorem, can be stated as follows (taken from Auslander and MacKenzie [1]):

THEOREM. *If $f_1, ..., f_s$ are differentiable functions on a neighborhood W of the point $x^0 = (x_1{}^0, ..., x_n{}^0) \in R^n$, if $f_1(x^0) = \cdots = f_s(x^0) = 0$ and if the $s \times n$ matrix*

$$J(x) = \begin{bmatrix} \dfrac{\partial f_1}{\partial x_1} & \cdots & \dfrac{\partial f_1}{\partial x_n} \\ \vdots & & \vdots \\ \dfrac{\partial f_s}{\partial x_1} & \cdots & \dfrac{\partial f_s}{\partial x_n} \end{bmatrix}$$

has constant rank r on W, with $r < n$, then there are a neighborhood U of $y^0 = (0, ..., 0) \in R^m$, where $m = n - r$ and a differentiable mapping $\eta: U \to W$ such that $\eta(y^0) = x^0$ and $f_1(\eta(y)) = \cdots = f_s(\eta(y)) = 0$ for $y \in U$. Let S be the variety defined by $f_1, ..., f_s$ in W. If W is sufficiently small, then there exists a differentiable mapping $\zeta: W \to R^m$ such that $\eta(U) = W \cap S$ and $\zeta \circ \eta$ is the identity on U.

[2] I am indebted to Alan Kirman for this reference. The results stated can be found in [7; problem 22, p. 38; answer on p. 156].

That is, the solutions to $f_1 = \cdots = f_s = 0$ form a set S which is diffeomorphic to R^m. Applied to the theorem in this paper, this says that the solutions to $g_i = 0$, $i = Tk - 2k + 1, \ldots, Tk$, form a manifold of dimension $Tk - k$. This follows from the fact that the rank of the Jacobian of the g_i is $2k$ and the g_i: $R^{Tk+k} \to R^{Tk}$; thus $n = Tk + k$ and $r = 2k$ in terms of the notation in the theorem as cited.

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