PRIMO: PRIVATE REGRESSION IN MULTIPLE OUTCOMES

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ABSTRACT

We introduce a new differentially private regression setting we call Private Regression in Multiple *Outcomes* (PRIMO), inspired the common situation where a data analyst wants to perform a set of lregressions while preserving privacy, where the covariates X are shared across all l regressions, and each regression $i \in [l]$ has a different vector of outcomes y_i . While naively applying private linear regression techniques l times leads to a \sqrt{l} multiplicative increase in error over the standard linear regression setting, in Subsection 4.1 we modify techniques based on sufficient statistics perturbation (SSP) to yield greatly improved dependence on l. In Subsection 4.2 we prove an equivalence to the problem of privately releasing the answers to a special class of low-sensitivity queries we call *inner* product queries. Via this equivalence, we adapt the geometric projection-based methods of [1] to the PRIMO setting. Under the assumption the labels Y are public, the projection gives improved results over the Gaussian mechanism when $n < l\sqrt{d}$, with no asymptotic dependence on l in the error. In Subsection 4.3 we study the complexity of our projection algorithm, and analyze a faster sub-sampling based variant in Subsection 4.4. Finally in Section 5 we apply our algorithms to the task of private genomic risk prediction for multiple phenotypes using data from the 1000 Genomes project. We find that for moderately large values of l our techniques drastically improve the accuracy relative to both the naive baseline that uses existing private regression methods and our modified SSP algorithm that doesn't use the projection.

1 Introduction

Linear regression is one of the most fundamental statistical tools used across the applied sciences, for both inference and prediction. In genetics, polygenic risk scores [2, 3] are computed by regressing phenotype (e.g. disease status) onto individual genomic data (SNPs) in order to identify genetic risk factors. In the social sciences, we might regress observed societal outcomes like income or marital status on a fixed set of covariates [4]. In many of these cases where the data records correspond to individuals, there are two aspects of the problem setting that co-occur:

- Aspect 1. The individuals may have a legal or moral right to privacy that has the potential to be compromised by their participation in a study.
- **Aspect 2.** Multiple regressions will be ran using the same set of individual characteristics across each regression with different outcomes, either within the same study or across many different studies.

Aspect 1 has been established as a legitimate concern through both theoretical and applied work. The seminal paper of [5] showed that the presence of an individual in a genomic dataset could be identified given simple summary statistics about the dataset, leading to widespread concern over the sharing of the results of genomic analyses. In the machine learning setting, where what is being released is a model w trained on the underlying data, there is a long line of research into "Membership Inference Attacks" [6, 7], which given access to w are able to identify which points are in the training set. Over the last decade, differential privacy [8] has emerged as a rigorous solution to the privacy risk posed by Aspect 1. In the particular case of linear regression the problem of how to privately compute the optimal regressor has been studied in great detail, which we summarise in Subsection 3.

Aspect 2 has been studied extensively from the orthogonal perspective of multiple hypothesis testing, but until now has not been considered in the context of privacy. The problem of overfitting or "p-hacking" in the social and natural

sciences has been referred to as the "statistical crisis in science" [9], and developing methods that quantify and mitigate the effects of overfitting has been the subject of much attention in the statistics and computer science communities [10, 11, 12]. Given the ubiquity of Aspects 1 and 2, this raises an important question:

When computing $i = 1 \dots l$ distinct regressions with a common set of X's and distinct y_i 's, what is the optimal accuracy-privacy tradeoff?

Interestingly, at a technical level, the problem of multiple hypothesis testing is related to differential privacy. It has been shown that if each query (in this case regression which is a special class of query called an optimization query [13]) is computed subject to differential privacy, then we can obtain a provable tradeoff between the number of noisy query answers we provide about a dataset and the extent of overfitting that is possible [10]. This gives a second motivation beyond privacy for our setting: even when the underlying data is not sensitive, our method can be viewed as a way to provably prevent overfitting when running multiple linear regressions with shared covariates.

1.1 Results

The primary contribution of this work is to introduce the novel PRIMO problem, and to provide a class of algorithms that trade off accuracy, privacy, and computation (Subsections 4.1, 4.2). In addition to introducing the PRIMO problem, to our knowledge we are the first to apply private query release methods to linear regression (Subsection 4.2). In Subsection 4.3 we analyze the computational complexity our method, and include an analysis of sub-sampled private linear regression (Subsection 4.4). We also provide a compelling practical application of our algorithms to the problem of genomic risk prediction with multiple phenotypes using data from the 1000 Genomes project [14].

Our Algorithm 2 crucially leverages the assumption that we are in the *public label* setting, where Y is considered public and X is considered private. In genomics a person's genome X is capable of revealing an as yet untold number of sensitive insights about their health, whereas the phenotypes (outcomes) y might be observable characteristics like height or eye color that are not private. We also remark that Algorithm 2 can just as easily be applied in the *private label* setting where X is considered public and Y is private, where the projection is into C(X) rather than C(Y) in Line 5. This setting is known as label differential privacy, and has been the subject of increasing interest [15, 16, 17]. Since the results from the public label setting encompass the results under label DP, with the added complexity of having to add noise to the covariance matrix $X^T X$, we state our results for the public label setting.

We will show that in a range of common parameter settings, our methods can obtain PRIMO at minimal cost. [18] shows that for private linear regression (PRIMO with l = 1), we have the following lower bound on the error $f(w_{private}) - f(w_{opt}) \ge$

$$\min\{||\mathcal{Y}||^2, \frac{\sqrt{d}(||\mathcal{X}||^2||\mathcal{W}||^2 + ||\mathcal{X}||||\mathcal{W}||||\mathcal{Y}||}{n\epsilon}\}$$
(1)

Since this lower bound holds for the case when l = 1, it of course holds for the PRIMO setting where l > 1. Throughout the paper this lower bound will serve as a benchmark for the cost to accuracy of taking l > 1, and in settings in which the bounds for PRIMO match this lower bound we will say we have "PRIMO for Free."

Theorems 4.1, 4.3 imply that if α is the difference between mean squared error of the private estimator output by Algorithm 1 with $\mathcal{M} = \text{GaussMech}$ and $\mathcal{M} = \text{Algorithm 2}$ respectively and the non-private OLS estimator, then with high probability:

$$\alpha^{2} = \tilde{O}(\frac{d||\mathcal{X}||^{4}||\hat{w}||^{4}}{n^{2}} + \min(\frac{||\hat{w}||^{2}\sqrt{d}|\mathcal{Y}|^{2}||\mathcal{X}||^{2}}{n}, \frac{||\hat{w}||^{2}ld|\mathcal{Y}|^{2}||\mathcal{X}||^{2}}{n^{2}}))$$
(2)

In Figure 1 we summarize the accuracy guarantees implied by Equation 2. When cost(l) = 1 this corresponds to "PRIMO for Free", and when $cost(l) = \sqrt{l}$ our upper bound matches that of the naive private baseline (Equation 6).

In Section 4.1 we adapt the previously proposed sufficient statistics perturbation (SSP) algorithm of [19, 20] to the PRIMO setting. Methods based on SSP rely on perturbing $X^T X \approx \Sigma$ and $X^T y \approx \Sigma_{Xy}$ separately, and then use these noisy estimates to compute the least squares estimator:

$$\tilde{w}_{i*} = \underbrace{(X^T X + E_1)^{-1}}_{\text{noisy covariance term }\tilde{\Sigma}} \times \underbrace{(X^T Y_i + E_{2i})}_{\text{noisy association term }\tilde{\Sigma}_{Xy}}$$
(3)

Via a novel accuracy analysis of SSP for the case when the privacy levels for E_1 , E_{2i} differ (Theorem 4.1), we show that since the noisy covariance matrix is reused across the regressions (as it only depends on X), by allocating the majority

PRIMO

Guarantees of ReuseCov				
# l of regressions	α	cost(l)	\mathcal{M}	Privacy
$l > \frac{n}{\sqrt{d}}, n < \frac{\sqrt{d} \mathcal{X} ^2 \mathcal{W} ^2}{ \mathcal{Y} ^2}$	$\left \begin{array}{c} \tilde{O}\left(\frac{\sqrt{d} \mathcal{X} ^2 \hat{w} ^2}{n} \right) \end{array} \right.$	1	Proj	Public Y
$l > \frac{n}{\sqrt{d}}, n > \frac{\sqrt{d} \mathcal{X} ^2 \mathcal{W} ^2}{ \mathcal{Y} ^2}$	$\left \tilde{O}\left(\frac{ \hat{w} d^{1/4} \mathcal{Y} \mathcal{X} }{\sqrt{n}}\right) \right.$	$\frac{n}{d^{1/4}}$	Proj	Public Y
$l < \min(\frac{n}{\sqrt{d}}, \frac{ \mathcal{X} ^2 \mathcal{W} ^2}{ \mathcal{Y} ^2})$	$\left \tilde{O}\left(\frac{\sqrt{d} \mathcal{X} ^2 \hat{w} ^2}{n}\right) \right.$	1	Gauss	Private X, Y
$l \in \left(\frac{ \mathcal{X} ^2 \mathcal{W} ^2}{ \mathcal{Y} ^2}, \frac{n}{\sqrt{d}}\right)$	$\left \tilde{O}\left(\frac{ \hat{w} \sqrt{ld} \mathcal{Y} \mathcal{X} }{n}\right) \right.$	\sqrt{l}	Gauss	Private X, Y

Figure 1: Here cost(l) is the ratio of the error for PRIMO given by Equation 2 to the lower bound in Equation 1, and \mathcal{M} denotes the mechanism used to compute $X^T Y$.

of our privacy budget to computing this term, we are able to obtain PRIMO for Free when $l < \min(\frac{n}{\sqrt{d}}, \frac{||\mathcal{X}||^2 ||\mathcal{W}||^2}{|\mathcal{Y}|^2})$. When *l* is sufficiently small, we obtain PRIMO for Free because the error term is dominated by the error in computing the noisy covariance matrix, which does not depend on *l*, rather than the error from computing the noisy association term.

Given that our PRIMO for Free result in Subsection 4.1 relies on taking l small enough that the error is dominated by the covariance error term, it is natural to ask if, under parameter regimes where the error from the association term dominates, if we can obtain improved dependence of α on l over the \sqrt{l} given by the Gaussian Mechanism. This is the focus of Subsection 4.2, where we start by showing that when we assume the labels Y are public, privately computing $\frac{1}{n}X^TY$ is equivalent to releasing dl "low sensitivity" queries [11] we call *inner product queries* (Definition 2.3). Our Algorithm 2 for privately answering these queries with low l_2^2 error is similar to the relaxed projection mechanism of [21], with a few key differences we elaborate on in Section 3. Via a refined analysis of this mechanism that exploits the geometric structure of our queries we obtain the surprising result that it is possible to completely remove the dependence of the error in computing the noisy association term on l and improve the dependence on d by a factor of $d^{1/4}$ – albeit at the cost of a factor of \sqrt{n} (Theorem 4.2).

Theorem 4.3 states the error for PRIMO of the variant of Algorithm 1 that uses this projection mechanism as a subroutine. Inspecting the accuracy bound, we see that when $l > \frac{n}{\sqrt{d}}$ but $\frac{n}{\sqrt{d}} < \frac{||\mathcal{X}||^2 ||\mathcal{W}||^2}{|\mathcal{Y}|^2}$ we also obtain PRIMO for Free. This is in contrast to when we obtained PRIMO for Free for small l in Subsection 4.1. In this case it is because (i) l is sufficiently large such that the error of the noisy association term when computed via Algorithm 2 is lower than when using the Gaussian Mechanism (Lemma 1) and (ii) n is sufficiently small such that the noisy covariance term dominates the error of the projection mechanism.

In Subsections 4.3, 4.4 we consider the computational complexity of our methods. In Subsection 4.3 we show that given the QR decomposition of the noisy covariance matrix and the SVD of the label matrix Y, standard techniques give a simple way to compute the private estimators for all l. When n > d > l the cost of SSP variants like our ReuseCov algorithm, are dominated by the cost of forming the covariance matrix, which is $O(nd^2)$ – prohibitively large for large n, d. To address this shortcoming, in Section 4.4 we develop a sub-sampling based version of ReuseCov which estimates the covariance matrix using a random sample of s points – reducing the computational cost to $O(sd^2)$.

2 Preliminaries

We start by defining the standard linear regression problem. Given $X \in \mathcal{X}^n \subset \mathbb{R}^{n \times d}, y_i \in \mathcal{Y}^n$, and parameter space \mathcal{W} , for $w \in \mathcal{W}$ let $f(w) := \frac{1}{n} \sum_{k=1}^n (w \cdot x_k - y_{ik})^2$ be the linear regression objective, and denote by $w_{i*} = \arg \min_w f(w) = (X^T X)^{-1} X^T y_i$ the ordinary least squares estimator (OLS). Let $f_\lambda(w) = \frac{1}{n} \sum_{k=1}^n (w \cdot x_k - y_{ik})^2 + \lambda ||w||_2^2$ the ridge regression objective, and let $w_{i*}^\lambda = \arg \min_w f_\lambda(w) = (\frac{1}{n} X^T X + \lambda I_d)^{-1} \frac{1}{n} X^T y_i$.

The definition of data privacy we use throughout is the popular (ε, δ) -differential privacy introduced in [22]. We refer the reader the to [8] for an overview of the basic properties of $(\varepsilon, \delta) - DP$ including closure under post-processing and advanced composition.

Definition 2.1. Let $\mathcal{M} : (\mathcal{X} \times \mathcal{Y})^n \to \mathcal{O}$ a randomized algorithm taking as input a dataset of n records. We say that $(X, Y) \sim (X', Y') \in (\mathcal{X} \times \mathcal{Y})^n$ if $\exists x, yx', y'$ such that $(X, Y) \cup \{x', y'\}/\{x, y\} = (X', Y')$. Then we say that \mathcal{M} is

 (ε, δ) -DP if $\forall (X, Y) \sim (X', Y'), o \subset \mathcal{O}$:

$$\Pr[\mathcal{M}(X,Y) \in o] \le e^{\varepsilon} \Pr[\mathcal{M}(X',Y') \in o] + \delta \tag{4}$$

In the less restrictive case where Equation 4 holds only over adjacent $X \sim X'$ with the same fixed Y, we say that we have differential privacy in the *public label* setting, which we consider in Section 4.2.

We now formalize the Private Regression In Multiple Outcomes (PRIMO) problem.

Definition 2.2. PRIMO. Let $x_i \in \mathcal{X} \subset \mathbb{R}^d$, $y_{ij} \in \mathcal{Y}$, for $i = 1 \dots n$, $j = 1 \dots l$. Let $X_{n \times d}$ the matrix with i^{th} row x_i , and let $Y_{n \times l}$ the matrix with j^{th} column $y_j = (y_{1j}, \dots, y_{nj})$. The optimal solution W^* to the PRIMO problem is

$$W^* = \inf_{W \in \mathcal{W}^l \subset \mathbb{R}^{d \times l}} ||XW - Y||_F^2$$

Given a randomized algorithm $\mathcal{M} : (\mathcal{X} \times \mathcal{Y}^l)^n \to \mathcal{W}^l$, we say that \mathcal{M} is an $(\alpha, \beta, \varepsilon, \delta)$ solution to the PRIMO problem if (i) \mathcal{M} is (ε, δ) -DP with respect to (X, Y) (or just X in the public label setting), and (ii) with probability $1 - \beta$ over $\tilde{W} \sim \mathcal{M}$:

$$l_{W}^{2}(X,Y) = \frac{1}{nl} ||X\tilde{W} - Y||_{F}^{2} - \frac{1}{nl} ||XW^{*} - Y||_{F}^{2} < \alpha$$

We will use y_j to denote the vector of n outcomes for the j^{th} outcome, and y^i for the vector of l outcomes corresponding to individual $i \in [n]$.

The $(\epsilon, \delta) - DP$ algorithm we will use most throughout is the Gaussian Mechanism. Note that we state a version of the Gaussian mechanism with constant $c(\epsilon, \delta)$ that is valid for all $\epsilon > 0$ (including $\epsilon > 1$), which follows from analyzing the mechanism using Renyi DP [23] and converting back to (ϵ, δ) -DP.

Lemma 1. GaussMech($\varepsilon, \delta, \Delta$) [8] Let $f : \mathcal{X}^n \to \mathbb{R}^d$ an arbitrary d-dimensional function, and define it's sensitivity $\Delta_2(f) = \sup_{X \sim X'} ||f(X) - f(X')||_2$, where $X \sim X'$ are datasets that differ in exactly one element. Then the Gaussian mechanism releases $f(X) + \mathcal{N}(0, \sigma^2)$, and is (ε, δ) -differentially private for $\sigma \ge c(\varepsilon, \delta)\Delta_2(f)$, where $c(\varepsilon, \delta) = \sqrt{2(\frac{1}{\epsilon} + \frac{\log \frac{1}{\delta}}{\varepsilon^2})}$.

Lastly, we define a new class of "low-sensitivity" queries [11] we call "inner product" queries that we will make heavy use of in Subsection 4.2.

Definition 2.3. Inner Product Query. Let $\mathcal{X} \subset \mathbb{R}^d$ and $X \in \mathcal{X}^n \subset \mathbb{R}^{d \times n}$ a dataset with column j corresponding to $x_j \in \mathcal{X}$. Let $y \in \mathbb{R}^n$, and $i \in [d]$. Then the pair (i, y) defines an inner product query $q_{(i,y)} : \mathcal{X}^n \to \mathbb{R}$,

$$q_{(i,y)}(X) := \frac{1}{n} \sum_{j=1}^{n} X_{ij} y_j = \frac{1}{n} e_i^T X y,$$

where e_i is the i^{th} basis vector in \mathbb{R}^d .

The l_2 sensitivity $\Delta_2(q_{(i,y)})$ is $\frac{||\mathcal{X}||_{\infty}|\mathcal{Y}|}{n}$, since changing one row of i of X changes at most one entry $\frac{1}{n}x_{ki}y_{ji}$ of $x_k^t y_i \leq \frac{||\mathcal{X}||_{\infty}|\mathcal{Y}|}{n}$.

Throughout the paper given a vector v and matrix A, ||v||, ||A|| denote the l_2 and the spectral norm respectively, $||A||_F$ is the Frobenius norm, $||\mathcal{X}|| = \sup_{x \in \mathcal{X}} ||x||_2$, and $||\mathcal{W}|| = \sup_{w \in \mathcal{W}} ||w||_2$.

3 Related Work

Private Linear Regression. Private linear regression is well-studied under a variety of different assumptions on the data generating distributions and parameter regimes. Typically analysis of private linear regression is done either under the fully agnostic setting where only parameter bounds $||\mathcal{X}||, |\mathcal{Y}|, ||\mathcal{W}||$ are assumed, under the assumption of a fixed design matrix and y generated by a linear Gaussian model (the so-called realizable case), or under the assumption of a random design matrix [24]. In this paper we focus on the first fully agnostic setting, because in our intended applications within the social and biomedical sciences in general we neither have realizability or Gaussian covariates. In the fully agnostic setting [25] provides a comprehensive survey of existing private regression approaches and bounds, including proposing a new adaptive technique.

Broadly speaking, techniques for private linear regression fall into 4 classes, sufficient statistics perturbation (SSP) [19, 20], Objective Perturbation (ObjPert) [26], Posterior sampling [27], and privatized (stochastic) gradient descent (NoisySGD) [28]. The methods in this paper are a sub-class of SSP-based methods, which correspond to Algorithm 1 where l = 1.

In the fully agnostic case, there are two regimes for private linear regression, each with upper bounds on empirical risk achieved by the algorithms above, and corresponding lower bounds stated below. The regimes depend on how well-conditioned the covariance matrix $X^T X$ is. Letting α^* be the inverse condition number we have the following lower bounds from [18]:

• When $\alpha^* \geq \frac{d^{1.5}(||\mathcal{X}||||\mathcal{W}||_{|}\mathcal{Y}|}{n||\mathcal{X}||||\mathcal{W}||\varepsilon},$ then:

$$f(\hat{w}) - f(w^*) \ge \min\{|\mathcal{Y}|^2, \frac{d^2(||\mathcal{X}||||\mathcal{W}|| + |\mathcal{Y}|)^2}{n^2 \alpha^* \varepsilon^2}\}$$

• We always have (even when $X^T X$ is ill-conditioned), $f(\hat{w}) - f(w^*) \ge 1$

$$\min\{|\mathcal{Y}|^2, \frac{\sqrt{d}(||\mathcal{X}||^2||\mathcal{W}||^2 + ||\mathcal{X}||||\mathcal{W}|||\mathcal{Y}|}{n\varepsilon}\}$$
(5)

Two techniques, NoisySGD and ObjPert both achieve the minimax lower bounds in both settings, although in order to achieve the minimax rates their hyperparameters depend on the unknown ||W|| and α^* , and so [25] proposes two adaptive methods that are based on Sufficient Statistics Perturbation, and are able to achieve optimal bounds in both settings. In Subsections 4.1,4.4 we state our theoretical results under the ill-conditioned setting as it is the most general, although analogous results hold in the well-conditioned setting as well.

Now given any (ε, δ) -DP algorithm for computing w_j privately, we can use it as a sub-routine to solve PRIMO by simply running it l times to compute each row of W. Hence by running any of the optimal algorithms l times with parameters $\varepsilon' \approx \varepsilon/\sqrt{l}, \delta' \approx \delta/l$, by advanced composition for differential privacy [8] we can achieve, subject to (ε, δ) -DP:

$$\alpha = \tilde{O}\left(\frac{\sqrt{ld}(||\mathcal{X}||^2 ||\mathcal{W}||^2 + ||\mathcal{X}||||\mathcal{W}|||\mathcal{Y}|}{n\varepsilon}\right)$$
(6)

So for a fixed privacy budget ε , this naive baseline is a factor of \sqrt{l} worse than in the standard private regression setting where l = 1.

Query Release. In Subsection 4.2 we show that privately computing the association term $\frac{1}{n}X^TY$ is equivalent to the problem of differentially privately releasing a set of $l \cdot d$ low-sensitivity queries [11]. While less is known about the optimal l_2 error for arbitrary low-sensitivity queries, it is clear that geometric techniques based on factorization and projection do not (at least obviously) apply, since there is no corresponding notion of the query matrix A_Q [29]. In the case where X, Y are both private, this corresponds to releasing a subset of $l \cdot d$ 2-way marginal queries over d + l dimensions, which is well-studied [30, 31, 32]. In the related work section of the Appendix we discuss why applying existing algorithms for private release of marginals seems unlikely be able to both improve over the Gaussian Mechanism and achieve non-trivial error. However, in the less restrictive but still practically relevant setting where the labels Y are public, computing the association term $\frac{1}{n}X^TY$ is equivalent to the problem of differentially privately releasing a special class of dl "low-sensitivity" queries we term "inner product queries" (Definition 2.3). For this special class of queries we can adapt the projected Gaussian Mechanism of [21] that is optimal for linear queries under l_2 error in the sparse regime. We are able to obtain greatly improved results over the naive Gaussian Mechanism in the regime where $l > \frac{n}{\sqrt{d}}$, as summarised in Figure 1 and presented in Subsection 4.2. This is a key regime of interest, particularly for genomic data, where d the number of SNPs could be in the hundreds of millions, n the population size in the hundreds of thousands ($\sim 400K$ for UK Biobank [33] one of the most popular databases), and l the number of recorded phenotypes could be in the thousands.

Projection mechanisms and Algorithm 2. Since we make heavy use of projection mechanisms in Subsection 4.2 in the *public label* setting, we elaborate on the difference between the existing methods and our Algorithm 2. The most technically related work to our Algorithm 2 is [21], which is itself a variant of the projection mechanism of [1]. There are several key differences in our analysis and application. The projection in [1] is designed for linear queries over a discrete domain, and runs in time polynomial in the domain size. Algorithm 2 allows continuous \mathcal{X}, \mathcal{Y} and runs in time polynomial in n, d, l. Like the relaxed projection mechanism [21], when our data is discrete we relax our data domain to be continuous in order to compute the projection more efficiently. Unlike in their setting which attempts to handle general linear queries, due to the linear structure of inner product queries our projection can be computed

in polynomial time via linear regression over the l_2 ball, as opposed to solving a possibly non-convex optimization problem. Moreover, due to this special structure we can use the same geometric techniques as in [1] to obtain theoretical accuracy guarantees (Theorem 4.2).

[1] bound the accuracy of their mechanism with respect to the *expected* mean squared error. Due to the necessity of analyzing the error of private regression with a high probability bound, rather than expected error, we prove a high probability error bound for the accuracy of the projection mechanism (Theorem 4.2). This requires application of the Hanson-Wright Inequality for anisotropic sub-Gaussian variables [34].

Further background on private query release for linear and low-sensitivity queries under l_{∞} and l_2 error is included in the Appendix. We also include further background on sub-sampled linear regression (Subsection 4.4) in the Appendix.

4 Methods

In Subsection 4.1 we develop our SSP-variant ReuseCov as a solution to the PRIMO problem. We give an error analysis of SSP for private linear regression where the noise levels for the covariance and association terms differ, and use this to examine when the asymptotic error of PRIMO matches the error of private linear regression ("PRIMO for Free"). In Subsection 4.2 we focus on the noisy association term, and give a reduction to the problem of private release of low-sensitivity queries. We show that for sufficiently large l, computing the association term via projection Algorithm 2 gives a polynomial improvement in error over the Gaussian mechanism when l is sufficiently large. In Subsection 4.3 we address the computational complexity of Algorithm 3. Finally, in Subsection 4.4 we address the high computational cost of computing the full covariance matrix $X^T X$ in $O(nd^2)$, and show how sub-sampling the covariance matrix allows us to trade-off computation with accuracy.

4.1 ReuseCov

 $\begin{array}{l} \hline \textbf{Algorithm 1 Input: } s, \lambda, \mathcal{X} \in \mathcal{X}^n \subset \mathbb{R}^{d \times n}, Y = [y_1, \ldots y_l] \in \mathcal{Y}^{l \times n}, \text{ privacy params: } \epsilon, \delta \\ \hline \lambda - \text{ReuseCov} \\ 1: \text{ Draw } E_1 \sim N_{d(d+1)/2}(0, \sigma_1^2), \text{ where } \sigma_1 = \frac{1}{s}c(\epsilon, \delta) ||\mathcal{X}||^2/\epsilon \\ 2: \text{ Compute } \hat{I} = (\frac{1}{n}X^TX + E_1 + \lambda I)^{-1} \\ 3: \text{ Draw } \hat{v} = [\hat{v}_1, \ldots \hat{v}_l] \sim \mathcal{M}(\epsilon/2, \delta/2, X, Y) \\ 4: \text{ for } i = 1 \ldots l \\ 5: \quad \text{Set } \hat{w}_i = \hat{I}\hat{v}_i \\ 6: \text{ end for} \\ 7: \text{ Return } \hat{W} = [\hat{w}_1, \ldots \hat{w}_l] \end{array}$

In the analysis of the ridge regression variant of SSP, which corresponds to Algorithm 1 with l = 1, Equation 13 in [25] shows w.p. $1 - \rho$:

$$f(\tilde{w}_{\lambda}) - f(w_{i*}) = \tilde{O}\left(\underbrace{\frac{d}{(\lambda + \lambda_{\min})\epsilon^2} ||\mathcal{X}||^2 |\mathcal{Y}|^2}_{\text{association error term}} + \underbrace{\frac{d}{(\lambda + \lambda_{\min})\epsilon^2} ||\mathcal{X}||^4 ||\mathcal{W}||^2}_{\text{covariance error term}}\right) + \underbrace{\frac{\lambda ||\mathcal{W}||^2}_{\text{error due to ridge penalty}}}_{\text{to variance error term}}$$
(7)

Inspecting these terms, we see that when $||\mathcal{X}||||\mathcal{W}|| \gg ||\mathcal{Y}||$ the error is dominated by the cost of privately computing $X^T X$, rather than privately computing $X^T y$. We now turn back to our PRIMO setting, and imagine independently applying SSP to solve each of our *l* regression problems. By the lower bounds in [18], given a fixed privacy budget this incurs at least a \sqrt{l} multiplicative blow up in error. However, we notice that when our private linear regression subroutine is an SSP variant, this naive scheme, of running *l* independent copies of SSP is grossly wasteful. Since the *X* matrix is shared across the different regression, we can simply compute our noisy estimate of $X^T X$ once, and then share that noisy covariance matrix across all of our regressions. Combining these two observations, we present Algorithm 1, where \mathcal{M} can be any ($\epsilon/2, \delta/2$)-DP algorithm for estimating $X^T Y$.

In line 2 we take $\epsilon_1 = \epsilon/2$, and $\epsilon_2 = \tilde{O}(\frac{\epsilon}{\sqrt{l}})$, allocating most of our privacy budget for the "harder" one-shot task of computing $X^T X$, and thus necessarily adding more noise to the association term. By advanced composition [8] this ensures (ϵ, δ) privacy overall, and propagating these noise terms into Equation 7 gives:

Theorem 4.1. With $\mathcal{M} = \text{GaussMech}(\epsilon/2, \delta/2, \Delta = \frac{1}{n}\sqrt{l}||\mathcal{X}|||\mathcal{Y}|)$, Algorithm 1 is an $(\alpha, \rho, \epsilon, \delta)$ solution to the PRIMO problem with

$$\alpha = \tilde{O}\left(||\hat{w}|| \sqrt{\frac{d||\mathcal{X}||^4 ||\hat{w}||^2}{n^2} + \frac{ld|\mathcal{Y}|^2 ||\mathcal{X}||^2}{n^2}}\right),\tag{8}$$

where $||\hat{w}||^2 = \frac{1}{l}||W^*||_F^2$, and \tilde{O} omits terms polynomial in $\frac{1}{\epsilon}, \log(1/\delta), \log(1/\rho)$.

While we defer the formal proof to the Appendix, the proof follows the analysis in [25] that uses Lemma 8 (essentially the Woodbury matrix inversion formula) to expand the difference $||\tilde{w}_{i*} - w_{i*}||_{X^T X}$ into the form in Equation 23 in the Appendix:

$$O\left(\underbrace{||E_1w_{i*}||^2_{(X^TX+\lambda I)^{-1}}}_{\text{covariance error}} + \underbrace{\lambda^2 ||w_{i*}||^2_{(X^TX+\lambda I)^{-1}}}_{\text{regularization error}}\right) + \tilde{O}\left(\underbrace{||E_2||^2_{(X^TX+\lambda I)^{-1}}}_{\text{association error}}\right)$$
(9)

The bound follows from using a Johnson-Lindenstrauss type lemma (Lemma 11) to bound the $||E_1||, ||E_2||$ terms, substituting in the noise levels in Algorithm 1, and optimizing over λ .

Inspecting Theorem 4.1, we see that when $l < \min(\frac{n}{\sqrt{d}}, \frac{||\mathcal{X}||^2 ||\mathcal{W}||^2}{|\mathcal{Y}|^2})$, Algorithm 1 with \mathcal{M} the Gaussian mechanism achieves error $\tilde{O}\left(\frac{\sqrt{d}||\mathcal{X}||^2 ||\hat{w}||^2}{n}\right)$. Since this matches the lower bound in Equation 1 for a single private regression in this case we say that we have achieved "PRIMO for Free."

4.2 Private Query Release

The improvement of Algorithm 1 over the naive PRIMO baseline in the previous sections make heavy use of the fact that the asymptotic error is dominated by the covariance error term. In the case when \mathcal{M} is the Gaussian mechanism, if we are in the regime where the association term dominates, ReuseCov still incurs a \sqrt{l} multiplicative factor in the error term, which does not improve over the baseline. In this section we show that via a reduction from computing the association term to private query release of inner product queries, under the weaker *public label* setting we can obtain improved bounds for PRIMO over the naive baseline and ReuseCov with $\mathcal{M} = \text{GaussMech when } l > \frac{n}{\sqrt{d}}$.

Let us reconsider the problem of privately computing the association term $\frac{1}{n}X^TY$ where $Y \in \mathcal{Y}^{n \times l}$, and $X \in \mathcal{X}^n \subset \mathbb{R}^{d \times n}$, and Y is considered public, so our data release needs only be DP in X. Consider entry (k, j) of $\frac{1}{n}X^TY$ for $k \in [d], j \in [l], a_{kj} = (\frac{1}{n}X^TY)_{kj} = e_k \frac{1}{n}X^Ty_j = q_{(k,y_j)}(X), q_{(k,y_j)}$ is the inner product query of Definition 2.3. Thus privately computing $\frac{1}{n}X^TY$, is equivalent to privately releasing answers to dl inner product queries $\{q_{(k,y_j)}(X)\}_{k \in [d], j \in [l]}$.

It will be convenient for us to write $q_{(k,y_j)}(X)$ as a single inner product. Let $\operatorname{vec}(X) = (x_{11}, \ldots, x_{1n}, x_{21}, \ldots, x_{d1}, \ldots, x_{dn}) \in \mathbb{R}^{nd}$, and given $y = \operatorname{vec}(X)$, let $\operatorname{mat}(y) = X$. Denote by $c_{kj} \in \mathbb{R}^{nd}$ the vector that has all zeros except in positions $(k-1)n+1, \ldots, kn$ it contains $\frac{1}{n}y_{1j}, \ldots, \frac{1}{n}y_{nj}$. Then it is clear that $q_{(k,y_j)}(X) = c_{kj}^T \operatorname{vec}(X)$, so if we let $C \in \{0,1\}^{dl \times dn}$ be the matrix with row $kj \in [dl]$ equal to c_{kj} , then $\frac{1}{n}X^TY = C \cdot \operatorname{vec}(X)$.

We now present our projection-based subroutine for privately computing $X^T Y$.

Algorithm 2 Input: $X \in \mathcal{X}^n \subset (\mathbb{R}^d)^n$, $Y = [y_1, \dots, y_l] \in \mathcal{Y}^{l \times n}$, privacy params: ϵ, δ ,

Inner Product Projection Mechanism

1: Formulate $C = C(Y) \in [0, 1]^{dl \times dn}$, vec(X). 2: Let $r = \frac{c(\epsilon, \delta) \sup_{i \in [l]} ||y^i||_2 \cdot ||\mathcal{X}||_2}{n}$ 3: Sample $w \sim N(0, 1)^{dl}$ 4: Let $\tilde{g} = Cvec(X) + rw$ 5: Let $\hat{g} = \operatorname{argmin}_{q \in K} ||g - \tilde{g}||_2^2$, where $K = C(n||\mathcal{X}||^2 B_1)$. Outputting \tilde{g} in Line 4 corresponds to the Gaussian Mechanism, and has mean squared error $r^2 = O(\frac{||\mathcal{Y}|||\mathcal{X}||^2}{n^2})$. Theorem 4.2 shows that the projection in Line 5 reduces error by a multiplicative factor of $\frac{n}{l\sqrt{d}}$.

Theorem 4.2. Let \mathcal{M} denote Algorithm 2. \mathcal{M} is (ϵ, δ) differentially private, and if $\hat{g} \sim \mathcal{M}$, then with probability $1 - \rho$:

$$\frac{1}{dl}||g - \tilde{g}||_2^2 = O\left(c(\epsilon, \delta)\sqrt{\log(2/\rho)}\frac{|\mathcal{Y}|^2||\mathcal{X}||^2}{n\sqrt{d}}\right)$$

Proof. \mathcal{M} is (ϵ, δ) differentially private by the Gaussian Mechanism and post-processing [8]. So we focus on the high probability accuracy bound. The crux of the proof is Lemma 2 which quantifies the reduction of error achieved by the projection step, and is the main workhorse behind the results in [1], but has been folklore in the statistics community since at least [35].

Lemma 2 ([35]). Let $K \subset \mathbb{R}^d$ be a symmetric convex body, let $g \in K$, and $\tilde{g} = g + w$ for some $w \in \mathbb{R}^d$. Then if $\hat{g} = \operatorname{argmin}_{g' \in K} ||\tilde{g} - g'||_2^2$, then

$$||\hat{g} - g||_2^2 \le \min\{4||w||_2^2, 4||w||_{K^\circ}\}$$

By Lemma 2 have that:

$$||\hat{g} - g||_2^2 \le 4||rw||_{K^\circ} = r \sup_{x \in K} x \cdot w \tag{10}$$

Since $K \subset (n||\mathcal{X}||^2)CB_1$, and using the fact that the l_2 norm is self-dual, we have:

$$||\hat{g} - g||_2^2 \le 4r||w||_{K^\circ} \le 4r||\mathcal{X}||\sqrt{n}\sup_{z\in B_1}(Cz)\cdot w = 4r||\mathcal{X}||\sqrt{n}||C^Tw||_2$$
(11)

So in order to bound $||\hat{g} - g||_2^2$ with high probability it suffices to bound $||C^T w||_2$ with high probability. This is the content of the Hanson-Wright Inequality for anisotropic random variables which we now state.

Lemma 3 (Hanson-Wright Inequality [34]). Let C^T an $m \times n$ matrix, and $z \sim \mathcal{N}(0, I_n) \in \mathbb{R}^n$. Then for a fixed constant c > 0:

$$\mathbb{P}[|||C^T z||_2 - ||C||_F| > t] \le 2e^{\frac{-ct^2}{||C||_F}}$$

Lemma 3 shows that $||C^T w||_2 = O(||C||_F \sqrt{\log(2/\rho)})$ with probability $1 - \rho$. Given $x_{kj} \in X$, let m = (k-1)n + j be the corresponding column of C, and let c^m denote this column. Then $||c^m||_2 = \frac{1}{n} ||(y_{1j}, \dots, y_{lj})||_2 = \frac{1}{n} ||y^j||_2 \le \frac{\sqrt{l}}{n}$. Hence $||C||_F = \frac{\sqrt{d\sum_{i=1}^n ||y^i||^2}}{n}$.

Plugging in the value of r gives, with probability $1 - \rho$:

$$\frac{1}{dl}\mathbb{E}[||\hat{g} - g||_2^2] = O\left(\frac{1}{dl} \cdot \sqrt{n}||\mathcal{X}||\frac{\sqrt{d\sum_{i=1}^n ||y^i||^2}}{n}\sqrt{\log(2/\rho)} \cdot \frac{c(\epsilon,\delta)||\mathcal{X}||\sup_i ||y^i||_2}{n}\right)$$
(12)

$$= O\left(\frac{c(\epsilon,\delta)\sqrt{\log(2/\rho)}\sqrt{\frac{1}{n}\sum_{i=1}^{n}||y^{i}||^{2}}\sup_{i}||y^{i}||_{2}||\mathcal{X}||^{2}}{nl\sqrt{d}}\right),$$
(13)

Since both terms involving Y in the numerator are $\leq \sqrt{l}|\mathcal{Y}|$ the bound follows. We note that since Y is public, in practice we can compute these terms rather than using $|\mathcal{Y}|$, the worst case bound.

We note that the mean squared error of the Gaussian mechanism without the projection is $O(r^2) = O(\frac{l||\mathcal{X}||^2|\mathcal{Y}|^2}{n^2})$, which for $l\sqrt{d} \gg n$ is strictly larger than the error of the projection mechanism. We also note that the bound in Theorem 4.3 is strictly better than the error given by applying the Median Mechanism algorithm of [36] for low-sensitivity queries, which is tailored for l_{∞} error, and which also requires discrete \mathcal{X} . For example, when $\mathcal{Y} = \{0, 1\}, \mathcal{X} = \{0, 1\}^d$, then $|\mathcal{Y}| = 1, ||\mathcal{X}|| = \sqrt{d}$, and so Theorem 4.2 gives a bound of $\tilde{O}(\frac{\sqrt{d}}{n})$, whereas the Median Mechanism gives $\tilde{O}(\frac{d^{2/3}\log(dl)^2}{n^{2/3}})$ for the mean squared error.

We now state the accuracy guarantees of Algorithm 1 with \mathcal{M} given by Algorithm 2.

Theorem 4.3. Let \mathcal{A} denote the label-private variant of Algorithm 1, where \mathcal{M} is Algorithm 2 with privacy parameters $(\epsilon/2, \delta/2)$. Then \mathcal{A} is an $(\alpha, \rho, \epsilon, \delta)$ solution to the PRIMO problem with

$$\alpha = \tilde{O}\left(||\hat{w}|| \sqrt{\frac{d||\hat{w}||^2(||\mathcal{X}||^4)}{n^2} + \frac{\sqrt{d}|\mathcal{Y}|^2||\mathcal{X}||^2}{n}}\right)$$

where $||\hat{w}||^2 = \frac{1}{l} ||W^*||_F^2$, and \tilde{O} omits terms polynomial in $\frac{1}{\epsilon}, \log(1/\delta), \log(1/\rho)$.

Inspecting Theorem 4.3 in the regime where the projection improves the error over the Gaussian Mechanim $l > \frac{n}{\sqrt{d}}$, when $n < \frac{\sqrt{d}||\mathcal{X}||^2||\mathcal{W}||^2}{|\mathcal{Y}|^2}$ is not too large, the dominant term in the error is $\tilde{O}\left(\frac{\sqrt{d}||\mathcal{X}||^2||\hat{w}||^2}{n}\right)$, and so we achieve PRIMO for Free. When n is sufficiently large the dominant error term is $\tilde{O}\left(\frac{||\hat{w}||d^{1/4}|\mathcal{Y}|||\mathcal{X}||}{\sqrt{n}}\right)$ which is a factor of $\frac{n}{d^{1/4}}$ worse than the lower bound.

4.3 Computational Complexity

The computational complexity of Algorithm 1 can be broken down into 3 components:

- Step 1: Forming $X^T X$, $(nd^2 \text{ or much faster})$
- Step 2: the cost of computing $\hat{I}^{-1}\hat{v}_i = (X^T X + \lambda I + E_1)^{-1}\hat{v}_i \ \forall i \in [l], (d^3 + ld^2)$
- Step 3: In the case where \mathcal{M} is the projection algorithm, computing the projection $\operatorname{argmin}_{v \in C(\mathcal{X}^n)} ||v \hat{v}_i||_2^2$, $(nl\min(n, l) + nd + nld$ via diagonalization)

Forming the covariance matrix $X^T X$ is a matrix multiplication of two $d \times n$ matrices, which can be done via the naive matrix multiplication in time $O(nd^2)$, and via a long-line of "fast" matrix multiplication algorithms in time $O(d^{2+\alpha(n)})$; For example if $n < d^{\cdot3}$ it can be done in time that is essentially $O(d^2)$ [37]. Step 2 can be completed by solving the equation $\hat{I}\hat{w}_i = \hat{v}_i, i = 1 \dots l$ via the conjugate gradient method, which takes time $O(\gamma(\hat{I})d^2\log(1/\epsilon))$ to compute an ϵ -approximate solution [38] where $\gamma(\hat{I})$ is the condition number. We note that this has to be done separately for each $i = 1 \dots l$ giving total time $O(l \cdot \gamma(\hat{I})d^2\log(1/\epsilon))$. Alternatively, an exact solution $\hat{I}^{-1}\hat{v}_i$ can be computed directly using the QR decomposition of the matrix \hat{I} . The decomposition $\hat{I} = QR$ can be computed in time $O(d^3)$ [38] and does not depend on the \hat{v}_i , after which using $R\hat{w}_i = Q^T \tilde{v}_i, \hat{w}_i$ can be computed in time $O(d^2)$ via backward substitution. This gives a total time complexity of $O(d^3 + ld^2)$. So if d and $\frac{1}{\gamma(\tilde{I})}$ are sufficiently small relative to l, e.g.

if $l = \tilde{\Omega}(d/(\gamma(\hat{I})))$, it will be faster to use the QR decomposition based method. The projection in Line 5 corresponds to minimizing a quadratic over a sphere. Setting $A = C^T C \in \mathbb{R}^{dn \times dn}$, $b = 2C^T \tilde{g} \in \mathbb{R}^{dl}$, then $\hat{v} = (Cn||\mathcal{X}||^2)x$, where $x \in \mathbb{R}^{nd}$ is the minimizer of:

$$\min_{x \in B_1} x^t A x - b^t x \tag{14}$$

s.t.
$$||x||_2 \le \sqrt{n} ||\mathcal{X}||$$
 (15)

Now, given the spectral decomposition of $A = U\Lambda U^T$, and the coordinates of b in the eigenbasis $U^T b$, Lemma 2.2 in [39] gives a simple closed form for x that computes each coordinate in constant time. Since there are nd coordinates of x, this incurs an additional additive factor of O(nd) in the complexity, which is dominated by the cost of diagonalizing A. So the complexity of this step is the complexity of diagonalizing $A = C^T C$, or equivalently finding the right singular vectors of C, plus the complexity of computing $U^T b$. This is seemingly bad news, as $C \in \mathbb{R}^{dn \times dl}$ is a very high-dimensional matrix, and the complexity for computing the SVD of C without any assumptions about its structure is $O(d^3 ln \min(l, n))$ [40]. However, it is evident from the construction of C in Subsection 4.2 that

$$C = I_d \otimes \frac{1}{n} Y^T,$$

where \otimes is the Kronecker product. Then if $L\Lambda V^T$ is the SVD of Y^T , standard properties of the Kronecker product imply that the spectral decomposition of $C^T C$ is:

$$C = \frac{1}{n} \cdot (I_d \otimes L)(I_d \otimes \Lambda)(I_d \otimes V^T) \implies (16)$$

$$C^{T}C = (I_{d} \otimes V)(I_{d} \otimes \Lambda^{2})(I_{d} \otimes V^{T})$$
(17)

Hence we can compute SPEC(C) in the time it takes to compute SVD(Y), or $O(nl\min(n, l))$. Similarly, to efficiently compute the $U^T b$ term required for Lemma 2.2 [39] we can again take advantage of properties of the Kronecker product, $U^T \dot{b} =$

$$(I_d \otimes V)^T 2C^T \tilde{g} = 2(I_d \otimes V^T)(I_d \otimes \frac{1}{n}Y)\tilde{g} = 2(I_d \otimes \frac{1}{n}V^T y)\tilde{g} = \operatorname{vec}(\frac{2}{n}V^T Y \operatorname{mat}(\tilde{g})),$$

where $mat(\tilde{g})$ is the $l \times d$ matrix with row *i* given by elements $(l(i-1)+1, \ldots, l(i-1)+l \text{ of } \tilde{g})$, and the last equality follows properties of the Kronecker product. Now $V^T Y = \Lambda U^T$ which can be computed in O(ln) since Λ is diagonal. Multiplying by $mat(\tilde{q})$ can be done in another O(nld), for total complexity of $O(nl \max(\min(n, l), d))$.

Putting the complexity of these steps together we get:

Theorem. The complexity of Algorithm 3 is $O(\max(\min(nl^2, n^2l), nld, nd^2, ld^2, d^3))$.

Algorithm 3 Input: $\lambda, \mathcal{X} \in \mathcal{X}^n \subset \mathbb{R}^{d \times n}, Y = [y_1, \dots, y_l] \in \mathcal{Y}^{l \times n}$, privacy params: ϵ, δ . We denote by \mathcal{B} the Algorithm in Lemma 2.2 [39]

 $\lambda - \texttt{ReuseCov}$ 1: Draw $E_1 \sim N_{d(d+1)/2}(0, \sigma_1^2)$, where $\sigma_1 = \frac{1}{n} 2\sqrt{2\log(2.5/\delta)} ||\mathcal{X}||^2/\epsilon$ 2: Compute $\hat{I} = (\frac{1}{n}X^TX + E_1 + \lambda I)$ 3: Compute the QR decomposition $\hat{I} = QR$ 4: Draw $\hat{v} = [\hat{v}_1, \dots, \hat{v}_l] \sim \text{GaussMech}(\epsilon/2, \delta/2, \Delta = \frac{1}{n}\sqrt{l}||\mathcal{X}|||\mathcal{Y}|)$ 5: Compute $SVD(Y^T) = U\Lambda V^T$ 6: Compute $\Pi_{C(Y)} \hat{v} = \mathcal{B}(V, \Lambda, \hat{v})$ 7: for i = 1 ... lSolve $R\hat{w}_i = Q^T \hat{v}_i$ by back substitution. 8: 9: end for 10: Return $\hat{W} = [\hat{w}_1, \dots, \hat{w}_l]$

4.4 Sub-sampling based ReuseCov

The discussion in the previous section shows that when n > d > l, the complexity of Algorithm 3 is $O(nd^2)$ or the cost of forming the covariance matrix. In this section we show how sub-sampling s < n points can improve this to $O(sd^2)$ by giving an analysis of sub-sampled SSP. The key ingredient is marrying the convergence of the sub-sampled covariance matrix to $X^T X$ with the accuracy analysis of SSP we saw in Section 4.1.

Algorithm 4 Input: $s, \lambda, \mathcal{X} \in \mathcal{X}^n \subset \mathbb{R}^{d \times n}, Y = [y_1, \dots, y_l] \in \mathcal{Y}^{l \times n}$, privacy params: ϵ, δ

 $\lambda - {\tt SubSampReuseCov}$

1: Sub-sample s points without replacement from \mathcal{X} , we denote the sub-sampled design matrix by X_S .

2: Let $(\epsilon_1, \delta_1) = (\frac{n}{s}\epsilon/2, \delta/2)$

3: Draw $E_1 \sim N_{d(d+1)/2}(0, \sigma_1^2)$, where $\sigma_1 = \frac{1}{n} 2\sqrt{2\log(2.5/\delta)} ||\mathcal{X}||^2/\epsilon_1$

4: Compute $\hat{I}_s = (\frac{1}{s}X_S^T X_S + E_1 + \lambda I)$

5: Draw $\hat{v} = [\hat{v}_1, \dots \hat{v}_l] \sim \text{GaussMech}(\epsilon/2, \delta/2, \Delta = \frac{1}{n}\sqrt{l}||\mathcal{X}|||\mathcal{Y}|)$ 6: for $i = 1 \dots l$ 7: Set $\hat{w}_i = \hat{I_s}^{-1} \hat{v}_i$

```
8: end for
```

9: Return $\hat{W} = [\hat{w}_1, \dots, \hat{w}_l]$

Our algorithm is based on the observation that if we sub-sample $S \subset [n], |S| = s$ points without replacement then:

- The cost of computing the covariance matrix $\Sigma_S = \sum_{k \in S} x_k x_k^T$ is $O(sd^2)$
- By the "secrecy of the sub-sample" principle [8], our privacy cost for estimating Σ_S is scaled down by a factor of s/n

• With high probability for sufficiently large $s, \Sigma_S \to \Sigma$ by a matrix-Chernoff bound for sampling without replacement [41]

Theorem. With $\mathcal{M} = \text{GaussMech}(\epsilon/2, \delta/2, \Delta = \frac{1}{n}\sqrt{l}||\mathcal{X}|||\mathcal{Y}|)$, Algorithm 4 is an $(\alpha, \rho, O(\epsilon), \delta)$ solution to the PRIMO problem with $\alpha^2 = 1$

$$O(||\hat{w}||^{2}||\mathcal{X}||^{2}|\mathcal{Y}|^{2}(\frac{\log(2d/\rho)}{s}) + \frac{d}{\lambda}(1 + \frac{\log(2d/\rho)}{s}) \cdot (\frac{||\mathcal{X}||^{4}n^{2}}{s^{4}\epsilon^{2}}||\mathcal{W}||^{2} + \frac{l||\mathcal{X}||^{2}||\mathcal{Y}||^{2}}{n^{2}\epsilon^{2}}))$$
(18)

Proof Sketch. Our analysis will hinge on the case where l = 1 e.g. that of standard private linear regression, which we will extend to the PRIMO case by our choice of ϵ as in the proof of Theorem 4.1. Now let:

- $w_{i*} = (X^T X)^{-1} X^T Y$ the least squares estimator as before
- $w_{i*}^{\lambda} = (\frac{1}{n}X^TX + \lambda I)^{-1}\frac{1}{n}X^TY$ the ridge regression estimator
- $w_s = (\frac{1}{s}X_S^T X_S + \lambda I)^{-1}(\frac{1}{n}X^T Y)$ the sub-sampled least squares estimator
- $\tilde{w}_s = (\frac{1}{s}X_S^T X_S + E_1 + \lambda I)^{-1}(\frac{1}{n}X^T Y + E_2)$ our differentially private estimate of w_s

We note that $0 \le f(w_{i*}^{\lambda}) - f(w_{i*}) \le \lambda \left(||w_{i*}||^2 - ||w_{i*}^{\lambda}||^2 \right) \le \lambda ||\mathcal{W}||^2$. Then by the Lemma 9 and Cauchy-Schwartz with respect to the norm $||\cdot||_{X^T X}$:

$$|f(\tilde{w}_{s}) - f(w_{i*})| = ||\tilde{w}_{s} - w_{i*}||_{\frac{2}{X^{T}X}}^{2} \leq 3||w_{i*} - w_{i*}||_{\frac{X^{T}X}{n}}^{2} + 3||w_{i*}^{\lambda} - w_{s}||_{\frac{X^{T}X}{n}}^{2} + 3||w_{s} - \tilde{w}_{s}||_{\frac{X^{T}X}{n}}^{2} \leq 3\lambda ||\mathcal{W}||^{2} + \underbrace{3||w_{s} - w_{i*}^{\lambda}||_{\frac{X^{T}X}{n}}^{2}}_{Matrix Chernoff} + \underbrace{3||\tilde{w}_{s} - w_{s}||_{\frac{X^{T}X}{n}}^{2}}_{SSP \text{ analysis+Matrix Chernoff}}$$
(19)

So it suffices to bound each term with high probability. The second term, $||\tilde{w}_s - w_s||_{X^T X + \lambda I}$ can be bounded using the same arguments as in Theorem 4.1, with small differences due to scaling. Crucially though, as we need to bound this in the norm induced by $X^T X$ rather than $X_S^T X_S$, we will need to utilize the convergence of $X_S^T X_S \to X^T X$ via Matrix-Chernoff bounds.

Lemma 4. Under the assumption $||\mathcal{X}|| = O(n\lambda)$, then w.p. $1 - \rho/2$:

$$||w_s - \tilde{w}_s||_{\frac{X^T X}{n}} = O(\frac{||X||^4}{n^2 \epsilon^2} ||\mathcal{W}||^2 \cdot \frac{d}{\lambda} \frac{\log(2d/\rho)}{s} + \frac{l||\mathcal{X}||^2 ||\mathcal{Y}||^2}{n^2 \epsilon^2} \cdot \frac{d}{\lambda} \frac{\log(2d/\rho)}{s})$$
(20)

Bounding the first term can be reduced to bounding $||I - (\frac{1}{n}X^TX)^{-1/2}(\frac{1}{s}X_S^TX_S)(\frac{1}{n}X^TX)^{-1/2}||_2$ which follows more directly via the Matrix-Chernoff bound for sub-sampling without replacement:

Lemma 5. Under the assumption $||\mathcal{X}|| = O(n\lambda)$, then w.p. $1 - \rho/2$:

$$||w_{i*}^{\lambda} - w_s||_{\frac{X^T X}{n}} = O\left(\frac{||\mathcal{X}|||\mathcal{Y}|\log(2d/\rho)}{\lambda s}\right)$$

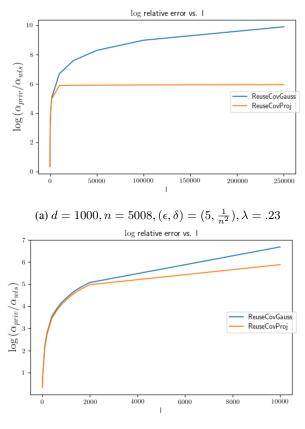
Substituting into Equation 20 and minimizing over λ gives the desired result.

5 Experiments

We implement Algorithm 1 with $\mathcal{M} = \text{GaussMech}$, and $\mathcal{M} = \text{ProjMech}$ (Algorithm 2), which we call ReuseCovGauss and ReuseCovProj respectively. We compare the two methods with each other and with SSP [19] across a range of parameters (d, l) covering each of the rows in Figure 1. Unsurprisingly, for even moderately large l, both variants of Algorithm 1 outperform naive SSP, whose accuracy degrades with \sqrt{l} . The more interesting question is fixing d, how large must we take l in practice such that ReuseCovGauss is outperformed by ReuseCovProj?

5.1 Genomic Data

Harkening back to our initial motivation to compute large numbers of polygenic risk scores with respect to a fixed genomic database, we perform our experiments using genotype data from the 1000 Genomes project [14]. The dataset contains 5008 haplotypes at 78961 SNPs. Throughout this section all experiments use all 5008 individuals. Whenever experiments are run with a fixed value of d < 78961, we have randomly sub-sampled d SNPs without replacement. In order to vary the number of phenotypes l we generate synthetic phenotype data using our haplotype dataset. After centering our haplotype matrix X by subtracting off the row means we generate synthetic haplotypes for $i = 1 \dots l$ by generating a random $\theta_i \sim \mathcal{N}(0, \frac{I_d}{\sqrt{d}}), y_{ij} \sim \theta_i \cdot x_j + \mathcal{N}(0, 1)$.



(b) Zoomed in to l < 10000

Figure 2: Comparing the log of the ratio of the squared loss of the private estimator to the square loss of the OLS estimator, across a range of l.

5.2 Results

Setting $(\epsilon, \delta) = (5, \frac{1}{n^2})$ with n = 5008, in Figure 2 we set d = 1000 and vary l = (1, 11, 101, 201, 401, 601, 801, 1001, 1201, 1401, 1601, 1801, 2000, 10000, 25000, 50000, 100000, 250000). Each value is averaged over 10 iterations for each value of <math>l. For large l, our results our well-supported by the theory; we observe that for l > 2000 ReuseCovProj begins to noticeably outperform ReuseCovGauss, and for very large l improves upon ReuseCovGauss by several orders of magnitude. In accordance with Theorem 4.3 for sufficiently large values of l the error of ReuseCovProj has no observable asymptotic dependence on l. Interestingly, by zooming into Figure 2(b) we see that the projection still performs as well as ReuseCovGauss even at small values of $l < \frac{n}{\sqrt{d}}$, whereas the analysis would suggest the projection could actually increase error relative to the Gaussian Mechanism in this parameter regime. One practical consequence of these experiments is that it appears the projection does not decrease accuracy even for small values of l, and at large values of l achieves a drastic improvement.

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6 Appendix

6.1 Additional Related Work

Private release of 2-way marginals Consider the problem of privately releasing all 2-way marginals over n points in $\{0, 1\}^{d+l}$. Theorem 5.7 in [30] gives a polynomial time algorithm based on relaxed projections that achieves mean squared error $\tilde{O}(n\sqrt{d+l})$, which matches the best known information theoretic upper bound [30], although there is a small gap to the existing lower bound $\min(n, (d+l)^2)$. This relaxed projection algorithm outperforms the Gaussian Mechanism when $n\sqrt{d+l} < ld \implies n < \frac{dl}{\sqrt{d+l}} \implies d\sqrt{d+l} > n$. Since the mean squared error of Algorithm 1 that used this projection as a subroutine is at least $\frac{d\sqrt{d+l}}{n}$, this means that in the regime where the projection outperforms the Gaussian Mechanism, we do not achieve mean squared error < 1 in our regression.

This suggests that, at least using the existing error analysis of SSP from [25], it seems unlikely that by applying specialized algorithms for private release of 2-way marginals to compute the $\frac{1}{n}X^TY$ term subject to differential privacy in (X, Y), e.g. the label private setting, we can both improve over the Gaussian Mechanism and achieve non-trivial error.

Linear Queries under l_{∞} -loss. Beyond 2-way marginals, the problem of privately releasing large numbers of linear queries (Definition 6.1) has been studied extensively. It is known that the worst case error is bounded by $\min(\frac{\sqrt{\log(|\mathcal{Q}|)}(\log|\mathcal{X}|\log(1/\delta))^{1/4}}{\sqrt{n\epsilon}}, \frac{\sqrt{|\mathcal{Q}|\log(1/\delta)}}{\epsilon n})$. The first term, which dominates in the so-called low-accuracy or "sparse" ([1]) regime, is achieved by the *PrivateMultiplicativeWeights* algorithm of [42], which is optimal over worst case workloads [43]. However, this algorithm has running time exponential in the data dimension, which is unavoidable [44] over worst case \mathcal{Q} . The second term, which dominates for $n \gg \frac{|\mathcal{Q}|}{\log |\mathcal{Q}|\log |\mathcal{X}|^{1/4}}$, the "high accuracy" regime, is achieved by the simple and efficient Gaussian Mechanism [8], which is also optimal over worst-case sets of queries \mathcal{Q} [43].

Linear Queries under l_2 -loss. For the l_2 error, in the high accuracy $n \gg |\mathcal{Q}|$ regime the factorization mechanism achieves error that is exactly tight for any workload of linear queries \mathcal{Q} up to a factor of $\log(1/\delta)$, although it is not efficient (Theorem [29]). In the low-accuracy regime, the algorithm of [1] that couples careful addition of correlated Gaussian noise (akin to the factorization mechanism) with an l_1 -ball projection step achieves error within log factors of what is (a slight variant) of a quantity known as the *hereditary discrepancy* $\operatorname{opt}_{\epsilon,\delta}(A, n)$ (Theorem [1]). This quantity is a known lower bound on the error of any (ϵ, δ) mechanism for answering linear queries [45], and so the upper bound is tight up to log factors in $|\mathcal{Q}|, |\mathcal{X}|$. Theorem 21 in [1] analyzes the simple projection mechanism that adds independent Gaussian noise and projects rather than first performing the decomposition step that utilizes correlated Gaussian noise, achieving error $O(nd \log(1/\delta) \sqrt{\log |X|}/\epsilon)$, which matches the best known (worst case over $\mathcal{Q})$ upper bound for the sparse n < d case [46]. In our Theorem 4.2 we give such a universal upper bound, rather than one that depends on the hereditary discrepancy of the matrix Y. While the bound can of course be improved for a specific set of outcomes Y by the addition of the decomposition step to the projection algorithm, we omit this step in favor of a simpler algorithm with more directly comparable bounds to existing private regression algorithms.

While the information-theoretic l_{∞} or l_2 error achievable for linear queries is well-understood [29, 1, 43], as synthetic data algorithms like *PrivateMultiplicativeWeights* and *MedianMechanism*, or the factorization or projection mechanisms are in general inefficient, there are many open problems pertaining to developing efficient algorithms for specific query classes, or heuristic approaches that work better in practice. Examples of these approaches along the lines of the factorization mechanism [47, 48], efficient approximations of the projection mechanism [21, 30], and using heuristic techniques from distribution learning in the framework of iterative synthetic data algorithms [49, 50, 51, 52].

Sub-sampled Linear Regression. In Subsection 4.4 we analyze SSP where we first sub-sample a random set of *s* points without replacement, and use this sub-sample to compute the noisy covariance matrix. Sub-sampled linear regression has been studied extensively absent privacy, where it is known that uniform sub-sampling is sub-optimal in that it produces biased estimates of the OLS estimator, and performs poorly in the presence of high-leverage points [53]. To address these shortcomings, techniques based on leverage score sampling [54], volume-based sampling [55][53], and spectral sparsification [56] have been developed. Crucially, in these methods the probability of a point being sub-sampled is data-dependent, and so they are (not obviously) compatible with differential privacy.

6.2 Lemmas and Definitions

Throughout the paper we make heavy use of common matrix and vector norms. For a vector $v \in \mathbb{R}^d$ and matrix $A \in \mathbb{R}^{d \times d}$, $||v||_A^2 := v^t A v$, $||A||_2^2 = \lambda_{\max}(A^T A)$, $||v||_2^2 = \sum_{k=1}^d v_k^2$, $||A||_F^2 = \sum_{i,j \in [d]} a_{ij}^2$.

Definition 6.1. Statistical Query [57] Let $D \in \mathcal{X}^n$ a dataset. A linear query is a function $q : \mathcal{X} \to [0,1]$, where $q(D) := \frac{1}{n} \sum_{x_i \in D}^n q(x_i)$.

Definition 6.2. [58] Let $\mathcal{X} = \{0, 1\}^m$, and $\mathcal{Q}_k = \{q_{ij}\}_{1 \le i_1 < i_2 < i_k \le m}$, where $q_{ij}(x) := \prod_{u=1}^k x_{i_u}$. Then the class \mathcal{Q}_k are called k-way marginals.

Lemma 6. [41] Let $Z_i \in \mathbb{S}_d^+$, and sample Z_1, \ldots, Z_s without replacement from $\{Z_1, \ldots, Z_n\}$. Suppose $\mathbb{E}[Z_i] = I_d$, and $\max_{i \in [n]} \lambda_{\max}(Z_i) \leq B$. Then for $\delta = \sqrt{\frac{2B \log(2d/\rho)}{s}}$, with probability $1 - \rho$:

$$\lambda_{\max}(\frac{1}{s}\sum_{i=1}^{s}Z_i) < 1+\delta, \ \lambda_{\min}(\frac{1}{s}\sum_{i=1}^{s}Z_i) > 1-\delta$$

Lemma 7 (folklore e.g. [59]). Given a dataset \mathcal{X}^n of n points and an (ϵ, δ) -DP mechanism M. Let the procedure subsample take a random subset of s points from \mathcal{X}^n without replacement. Then if $\gamma = s/n$, the procedure $M \circ$ subsample is $(O(\gamma \epsilon), \gamma \delta)$ -DP for sufficiently small ϵ .

6.3 **Proofs from Subsection 4.1**

The following lemma is used repeatedly in analyzing the accuracy of all SSP variants.

Lemma 8. Let A, B invertible matrices in $\mathbb{R}^{n \times n}$, and v, c vectors $\in \mathbb{R}^{n}$.

Then

$$A^{-1}v - (A+B)^{-1}(v+c) = (A+B)^{-1}BA^{-1}v - (A+B)^{-1}c$$

Proof. Expanding we get that:

$$A^{-1}v - (A+B)^{-1}(v+c) = (A^{-1} - (A+B)^{-1})v - (A+B)^{-1}c,$$

so it suffices to show that:

$$(A^{-1}-(A+B)^{-1})v=(A+B)^{-1}BA^{-1}v$$
 Now the Woodbury formula tells us that $(A+B)^{-1}=A^{-1}-(A+AB^{-1}A)^{-1},$ hence

$$(A^{-1} - (A + B)^{-1})v = (A + AB^{-1}A)^{-1}v = (A(I + B^{-1}A))^{-1}v$$

Then since:

$$(A(I+B^{-1}A))^{-1} = (I+B^{-1}A)^{-1}A^{-1} = (B^{-1}(B+A))^{-1A^{-1}} = (B+A)^{-1}BA^{-1},$$

we are done.

Proof of (ϵ, δ) **-DP in Theorem 4.1**:

Proof. The privacy proof follows from a straightforward application of the Gaussian mechanism. We note that releasing each \hat{v}_i privately, is equivalent to computing $X^TY + E_2$, where $E_2 \sim N_{d \times l}(0, \sigma_2^2)$. Now it is easy to compute *l*-sensitivity $\Delta(f)$ of $f(X) = X^TY$. Fix an individual *i*, and an adjacent dataset $X' = X/\{x_i, y_i\} \cup \{x'_i, y'_i\}$. Then $f(X) - f(X') = \Delta_V = [y_{i1}x_i - y'_{i1}x'_{i1}, \dots y_{il}x_i - y'_{il}x'_{il}]$. Then:

$$||f(X) - f(X')|| = \sqrt{||\Delta_V||_F^2} = \sqrt{\sum_{j=1}^l ||y_{ij}x_i - y'_{ij}x'_{ij}||^2} \le \sqrt{l \cdot 4||\mathcal{X}||^2||\mathcal{Y}||^2} = 2\sqrt{l}||\mathcal{X}||||\mathcal{Y}||^2$$

Hence setting $\sigma_2 = c(\epsilon, \delta) 2\sqrt{l} ||\mathcal{X}|| ||\mathcal{Y}|| / \epsilon$ by the Gaussian mechanism [8] publishing \hat{V} satisfies $(\epsilon/2, \delta/2) - DP$. Similarly if $g = X^T X$, $\Delta(g) \leq ||\mathcal{X}||^2$, and so setting $\sigma_1 = c(\epsilon, \delta) ||\mathcal{X}||^2 / \epsilon$, means publishing \hat{I} is $(\epsilon/2, \delta/2)$ -DP. By basic composition for DP, the entire mechanism is (ϵ, δ) -DP.

Proof of Accuracy in Theorem 4.1:

Proof. We follow the general proof technique developed in [25] analysing the accuracy guarantees of the ridge regression variant of SSP in the case $\lambda_{\min}(X^T X) = 0$, adding some mathematical detail to their exposition, and doing the appropriate book-keeping to handle our setting where the privacy level (as a function of the noise level) guaranteed by E_1 and E_2 differ. The reader less interested in these details can skip to Equation 25 below for the punchline.

Fix a specific index $i \in [l]$, and let $y = y_i$. We will analyze the prediction error of \hat{w}_i e.g. $F(w_i) - F(w_i^*)$. Then the following result is stated in [25] for which provide a short proof:

Lemma 9.

$$F(\hat{w}_i) - F(w_{i*}) = ||y - X\hat{w}_i||^2 - ||y - Xw_{i*}|| = ||\hat{w}_i - w_{i*}||_{X^T X}^2$$

Proof. We note that all derivatives of orders higher than 2 of $f(w) = ||y - Xw||^2$ are zero, and that $\nabla f_{w_{i*}} = 0$ by the optimality of w_{i*} . We also note that the Hessian $\nabla^2 f_w = X^T X$ at all points w. Then by the Taylor expansion of f(w) around w_{i*} :

$$f(\hat{w}_{i}) = f(w_{i*}) + (\hat{w}_{i} - w_{i*}) \cdot \nabla f_{w_{i*}} + (\hat{w}_{i} - w_{i*})' X^{T} X(\hat{w}_{i} - w_{i*})$$

0 and rearranging terms gives the result.

Which using $\nabla f_{w_{i*}} = 0$ and rearranging terms gives the result.

Now Corollary 7 in the Appendix of [25] states (without proof) the below identity, which we provide a proof of for completeness via Lemma 8:

$$\hat{w}_i - w_{i*} = (-X^T X + \lambda I + E_1)^{-1} E_1 w_{i*} - \lambda (X^T X + \lambda I + E_1)^{-1} w_{i*} + (X^T X + \lambda I + E_1)^{-1} E_2$$
(21)

Hence, still following [25], for any psd matrix A, $||\hat{w}_i - w_{i*}||_A^2 \leq$

$$3||(X^TX + \lambda I + E_1)^{-1}E_1w_{i*}||_A^2 + 3\lambda^2||(X^TX + \lambda I + E_1)^{-1}w_{i*}||_A^2 + 3||(X^TX + \lambda I + E_1)^{-1}E_2||_A^2$$
(22)
Lemma 10. [25] With probability $1 - \rho$, $||E_1|| \le (\lambda_{\min}(X^TX) + \lambda)/2$, and hence

$$X^T X + \lambda I + E_1 \succ .5(X^T X + \lambda I)$$

We also remark that $||By||_A^2 = (By)^T A By = ||y||_{B^T A B}^2$ for any vector y, and matrices A, B. Hence, Inequality 22, with $A = X^T X$ can be further simplified to:

$$3||(X^{T}X + \lambda I + E_{1})^{-1}E_{1}w_{i*}||_{A}^{2} + 3\lambda^{2}||(X^{T}X + \lambda I + E_{1})^{-1}w_{i*}||_{A}^{2} + 3||(X^{T}X + \lambda I + E_{1})^{-1}E_{2}||_{A}^{2} \leq O\left(||E_{1}w_{i*}||_{(X^{T}X + \lambda I + E_{1})^{-1}}^{2} + \lambda^{2}||w_{i*}||_{(X^{T}X + \lambda I + E_{1})^{-1}}^{2} + ||E_{2}||_{(X^{T}X + \lambda I + E_{1})^{-1}}^{2}\right) \leq (\text{Lemma 10}) O\left(||E_{1}w_{i*}||_{(X^{T}X + \lambda I)^{-1}}^{2} + \lambda^{2}||w_{i*}||_{(X^{T}X + \lambda I)^{-1}}^{2} + ||E_{2}||_{(X^{T}X + \lambda I)^{-1}}^{2}\right)$$
(23)

By basic properties of the trace we have: $\operatorname{tr}((\lambda I + X^T X)^{-1}) \leq d\lambda_{max}(\lambda I + X^T X^{-1}) = \frac{d}{\lambda_{\min}(\lambda I + X^T X)} = \frac{d}{(\lambda_{\min} + \lambda)}$, and $||w_{i*}||^2_{(X^T X + \lambda I)^{-1}} \leq \frac{||w_{i*}||^2}{\lambda}$. Continuing from [25] by their Lemma 6, we can bound each $||E_1 w_{i*}||^2_{(X^T X + \lambda I)^{-1}}$ and $||E_2||^2_{(X^T X + \lambda I)^{-1}}$.

Lemma 11 ([25]). Let $\theta \in \mathbb{R}^d$ and let E a symmetric Gaussian matrix where the upper triangular region is sampled from $N(0, \sigma^2)$ and let A be any psd matrix. Then with probability $1 - \rho$:

$$||E\theta||_A^2 \le \sigma^2 tr(A) ||\theta||^2 \log(2d^2/\rho)$$

Then recalling that:

• $\sigma_1^2 = \tilde{O}(||\mathcal{X}||^4/\epsilon^2)$ • $\sigma_2^2 = \tilde{O}(l||\mathcal{X}||^2|\mathcal{Y}|^2/\epsilon^2)$

Plugging into Lemma 11, and bringing it all together we get:

$$|F(\hat{w}_{i}) - F(w_{i*})| \leq ||w_{i*} - \hat{w}_{i}||_{X^{T}X} = O\left(||E_{1}w_{i*}||^{2}_{(X^{T}X + \lambda I)^{-1}} + \lambda^{2}||w_{i*}||^{2}_{(X^{T}X + \lambda I)^{-1}} + ||E_{2}||^{2}_{(X^{T}X + \lambda I)^{-1}}\right) = \tilde{O}\left(\frac{d}{\lambda_{\min} + \lambda}||w_{i*}||^{2}(||\mathcal{X}||^{4}/\epsilon^{2})\log(2d^{2}/\rho) + \lambda||w_{i*}||^{2} + \frac{d}{\lambda_{\min} + \lambda}l||\mathcal{X}||^{2}|\mathcal{Y}|^{2}/\epsilon^{2}\log(2d^{2}/\rho)\right)$$
(24)

Upper bounding Equation 24 by taking $\lambda_{\min} = 0$, we minimize over λ setting

$$\lambda = \tilde{O}\left(\frac{1}{\epsilon}\sqrt{d\log(2d^{2}/\rho)}||X||\sqrt{||\mathcal{X}||^{2} + \frac{l|\mathcal{Y}|^{2}}{||w_{i*}||^{2}}}\right) \Longrightarrow |F(\hat{w}_{i}) - F(w_{i*})| = \tilde{O}\left(\frac{1}{\epsilon}\sqrt{d\log(2d^{2}/\rho)}||X||\sqrt{||\mathcal{X}||^{2}||w_{i*}||^{4} + l|\mathcal{Y}|^{2}||w_{i*}||^{2}}\right)$$
(25)

Now if we are in the η -small regime, we have $|\mathcal{Y}| \leq \eta ||\mathcal{X}||||w_{i*}||$, and so

$$|l|\mathcal{Y}|^2 w_{i*}^2 \le l\eta^2 ||\mathcal{X}||^2 ||w_{i*}||^4,$$

which reduces Equation 25 to:

$$|F(\hat{w}_i) - F(w_{i*})| = \tilde{O}\left(\frac{1}{\epsilon}\sqrt{d\log(2d^2/\rho)}||X||^2||w_{i*}||^2(\eta\sqrt{l})\right),$$

as desired.

Proof of Theorem 4.3

Proof. We start with our usual expansion of $f(\hat{w}_i) - f(w_{i*})$, up until Equation 24, we have with probability $1 - \rho$ for every $i \in [l]$:

$$n \cdot f(\hat{w}_i) - f(w_{i*}) = \tilde{O}\left(\frac{d}{\lambda_{\min} + \lambda} ||w_{i*}||^2 (||\mathcal{X}||^4/\epsilon^2) \log(2d^2/\rho) + \lambda ||w_{i*}||^2 + \frac{1}{\lambda_{\min} + \lambda} ||E_{2i}||_2^2\right)$$
(26)

Aggregating over *i* and rearranging gives:

$$\frac{n}{l}\sum_{i=1}^{l}f(\hat{w}_{i}) - f(w_{i*}) = \tilde{O}\left(\frac{d}{\lambda_{\min} + \lambda}||\hat{w}||^{2}(||\mathcal{X}||^{4}/\epsilon^{2})\log(2d^{2}/\rho) + \lambda||\hat{w}||^{2} + \frac{1}{\lambda_{\min} + \lambda}\frac{1}{l}\sum_{i=1}^{l}||E_{2i}||_{2}^{2}\right), \quad (27)$$

where $||\hat{w}||^2 = \frac{1}{l} \sum_{i=1}^{l} ||w_{i*}||^2 = \frac{1}{l} ||W^*||_F^2$. Then by Theorem 4.2, we have with $\frac{1}{l} \sum_{i=1}^{l} ||E_{2i}||_2^2 = O\left(c(\epsilon, \delta)\sqrt{\log(2/\rho)}n\sqrt{d}|\mathcal{Y}|^2||\mathcal{X}||^2\right)$ with probability $1 - \rho$. So with probability $1 - 2\rho$, we have $\frac{n}{l} \sum_{i=1}^{l} f(\hat{w}_i) - f(w_{i*}) =$

$$\tilde{O}\left(\frac{d}{\lambda_{\min}+\lambda}||\hat{w}||^{2}(||\mathcal{X}||^{4}/\epsilon^{2})\log(2d^{2}/\rho)+\lambda||\hat{w}||^{2}+\frac{1}{\lambda_{\min}+\lambda}c(\epsilon,\delta)\sqrt{\log(2/\rho)}n\sqrt{d}|\mathcal{Y}|^{2}||\mathcal{X}||^{2}\right)$$
(28)

Finally optimizing over λ gives the desired result:

$$\alpha = \tilde{O}\left(||\hat{w}|| \sqrt{\frac{d||\hat{w}||^2 (||\mathcal{X}||^4/\epsilon^2) \log(2d^2/\rho)}{n^2}} + \frac{c(\epsilon, \delta)\sqrt{\log(2/\rho)}\sqrt{d}|\mathcal{Y}|^2||\mathcal{X}||^2}{n}\right)$$

6.4 Proofs from Subsection 4.4

Proof of Theorem 4.4:

Proof. Our analysis will hinge on the case where l = 1 e.g. that of standard private linear regression, which we will extend to the PRIMO case by our choice of ϵ as in the proof of Theorem 4.1. The fact that the Algorithm is $(O(\epsilon), \delta)$ private follows immediately from the Gaussian mechanism, and the secrecy of the sub-sample lemma (Lemma 7), which is why we can set $\epsilon_1 = \frac{n}{s} \epsilon/2$ in Line 2. We proceed with the accuracy analysis.

Define:

- $w_{i*} = (X^T X)^{-1} X^T Y$: the least squares estimator as before
- $w_{i*}^{\lambda} = (\frac{1}{n}X^TX + \lambda I)^{-1}\frac{1}{n}X^TY$: the ridge regression estimator
- $w_s = (\frac{1}{s}X_S^T X_S + \lambda I)^{-1}(\frac{1}{n}X^T Y)$: the sub-sampled least squares estimator
- $\tilde{w}_s = (\frac{1}{s}X_S^T X_S + E_1 + \lambda I)^{-1}(\frac{1}{n}X^T Y + E_2)$: differentially private estimate of w_s

We note that $0 \le f(w_{i*}^{\lambda}) - f(w_{i*}) \le \lambda \left(||w_{i*}||^2 - ||w_{i*}^{\lambda}||^2 \right) \le \lambda ||\mathcal{W}||^2$. Then by the Lemma 9 and Cauchy-Schwartz with respect to the norm $|| \cdot ||_{X^T X}$:

$$|f(\tilde{w}_{s}) - f(w_{i*})| = ||\tilde{w}_{s} - w_{i*}||_{\frac{X^{T}X}{n}}^{2} \leq 3||w_{i*} - w_{i*}^{\lambda}||_{\frac{X^{T}X}{n}}^{2} + 3||w_{i*}^{\lambda} - w_{s}||_{\frac{X^{T}X}{n}}^{2} + 3||w_{s} - \tilde{w}_{s}||_{\frac{X^{T}X}{n}}^{2} \leq 3\lambda ||\mathcal{W}||^{2} + 3||w_{s} - w_{i*}^{\lambda}||_{\frac{X^{T}X}{n}}^{2} + 3||\tilde{w}_{s} - w_{s}||_{\frac{X^{T}X}{n}}^{2}$$
(29)

Lemmas 12, 13 bound these terms with high probability.

Lemma 12. Under the assumption $||\mathcal{X}|| = O(n\lambda)$, then w.p. $1 - \rho/2$:

$$||w_{i*}^{\lambda} - w_s||_{\frac{X^T X}{n}} = O\left(\frac{||\mathcal{X}|||\mathcal{Y}|\log(2d/\rho)}{\lambda s}\right)$$

Lemma 13. Under the assumption $||\mathcal{X}|| = O(n\lambda)$, then w.p. $1 - \rho/2$:

$$||w_s - \tilde{w}_s||_{\frac{X^T X}{n}} = O(\frac{||X||^4}{n^2 \epsilon^2} ||\mathcal{W}||^2 \cdot \frac{d}{\lambda} \frac{\log(2d/\rho)}{s} + \frac{l||\mathcal{X}||^2 ||\mathcal{Y}||^2}{n^2 \epsilon^2} \cdot \frac{d}{\lambda} \frac{\log(2d/\rho)}{s})$$
(30)

Then by Equation 29 and Lemmas 12, 13, we get that w.p. $1 - \rho$:

$$f(\tilde{w}_{s}) - f(w_{i*}) = O\left(\lambda ||w_{i*}||^{2} + \frac{||\mathcal{X}||\mathcal{Y}|}{\lambda} (\frac{\log(2d/\rho)}{s}) + \frac{||X||^{4}n^{2}}{s^{4}\epsilon^{2}} ||\mathcal{W}||^{2} \cdot \frac{d}{\lambda} (1 + \frac{\log(2d/\rho)}{s}) + \frac{l||\mathcal{X}||^{2}||\mathcal{Y}||^{2}}{n^{2}\epsilon^{2}} \cdot \frac{d}{\lambda} (1 + \frac{\log(2d/\rho)}{s})\right),$$
(31)

Summing over i and minimizing over λ we set

$$\lambda = \frac{\sqrt{||\mathcal{X}|||\mathcal{Y}|(\frac{\log(2d/\rho)}{s}) + \frac{||X||^4 n^2}{s^4 \epsilon^2} ||\mathcal{W}||^2 \cdot \frac{d}{\lambda} (1 + \frac{\log(2d/\rho)}{s})} + \frac{l||\mathcal{X}||^2 ||\mathcal{Y}||^2}{n^2 \epsilon^2} \cdot \frac{d}{\lambda} (1 + \frac{\log(2d/\rho)}{s})}{\sqrt{1} l||W||_F},$$

which completes the result.

Proof of Lemma 12:

Proof. Now:

$$||w_{i*}^{\lambda} - w_{s}||_{\frac{1}{n}X^{T}X}^{2} = n||w_{i*}^{\lambda} - w_{s}||_{\frac{1}{n}X^{T}X}^{2} \le n||w_{i*}^{\lambda} - w_{s}||_{\frac{1}{n}X^{T}X+\lambda I}^{2}$$

We will focus on $||w_{i*}^{\lambda} - w_s||_{\frac{1}{n}X^TX + \lambda I}^2$. Let $\Sigma = \frac{1}{n}X^TX + \lambda I$, $\Sigma_s = \frac{1}{s}\sum_{j \in S} x_s x_s^T + \lambda I$, and $v = \frac{1}{n}X^TY$. Expanding $||w_{i*}^{\lambda} - w_s||_{\Sigma} =$

$$(\Sigma_{s}^{-1}v - \Sigma^{-1}v)^{T}\Sigma(\Sigma_{s}^{-1}v - \Sigma^{-1}v) = v^{T}\left(\overline{\Sigma_{s}^{-1} - \Sigma^{-1}}\right)\Sigma(\Sigma_{s}^{-1} - \Sigma^{-1}) v = v^{T}Av$$
(32)

Now since A is Hermitian, we know $||v||_A \leq ||v||||A||_2$. Since $||v|| \leq ||\mathcal{X}|||\mathcal{Y}|$, it suffices to bound $||A||_2$ with high probability. Noting that $A = (\Sigma_s^{-1} - \Sigma^{-1})\Sigma(\Sigma_s^{-1} - \Sigma^{-1}) = \Sigma_s^{-1}\Sigma^{1/2}(I - \Sigma^{-1/2}\Sigma_s\Sigma^{-1/2})(I - \Sigma^{-1/2}\Sigma_s\Sigma^{-1/2})\Sigma_s^{-1/2}\Sigma_s\Sigma^{-1/2})$, we have by the sub-multiplicativity of the operator norm:

$$||A||_{2} \leq \left(||\Sigma_{S}^{-1}\Sigma^{1/2}||_{2}^{2}\right) \cdot \left(||I - \Sigma^{-1/2}\Sigma_{s}\Sigma^{-1/2}||_{2}^{2}\right) \leq \frac{\lambda_{\max}(\Sigma)}{\lambda^{2}} \cdot \left(||I - \Sigma^{-1/2}\Sigma_{s}\Sigma^{-1/2}||_{2}^{2}\right)$$
(33)

Now consider $\Sigma^{-1/2} \Sigma_s \Sigma^{-1/2} = \Sigma^{-1/2} \frac{1}{s} \sum_{j \in S} (x_s x_s^T + \lambda I) \Sigma^{-1/2} = \frac{1}{s} \sum_{j \in S} Z_i$. Then note that $\mathbb{E}[Z_i] = \Sigma^{-1/2} (\frac{1}{n} \mathbb{E}[x_i x_i^T] + \lambda I) \Sigma^{-1/2} = \Sigma^{-1/2} \Sigma \Sigma^{-1/2} = I$, and that $\lambda_{max}(Z_i) \leq ||\Sigma^{-1}||_2 ||\frac{1}{n} x_i x_i^T + \lambda I||_2 \leq 1 + \frac{||\mathcal{X}||}{n\lambda}$. Now we can bound $||I - \frac{1}{s} \sum_{j \in S} Z_i||_2$ by Theorem 2.2 in [41]:

Lemma 14. [41] Let Z_1, \ldots, Z_s sampled without replacement from $\{Z_1, \ldots, Z_n\}$. Then if $Z_i \in \mathbb{S}_d^+, \mathbb{E}[Z_i] = I_d$, and $\max_{i \in [n]} \lambda_{\max}(Z_i) \leq B$ w.p. $1 - \rho$, for $\delta = \sqrt{\frac{2B \log(2d/\rho)}{s}}$:

$$\lambda_{\max}(\frac{1}{s}\sum_{i=1}^{s}Z_i) < 1+\delta, \ \lambda_{\min}(\frac{1}{s}\sum_{i=1}^{s}Z_i) > 1-\delta$$

So by Lemma 14, we know that with probability $1 - \rho$: $|\lambda_{\min}(I - \Sigma^{-1/2}\Sigma_s\Sigma^{-1/2})| = 1 - \lambda_{\max}(\Sigma^{-1/2}\Sigma_s\Sigma^{-1/2}) \le \delta$, and similarly $|\lambda_{\max}(I - \Sigma^{-1/2}\Sigma_s\Sigma^{-1/2})| \le \delta$, thus $||I - \Sigma^{-1/2}\Sigma_s\Sigma^{-1/2})||_2 \le \delta$. Substituting this all into Equation 33 and noting $\lambda_{\max}(\Sigma) \le \frac{||\mathcal{X}||}{n} + \lambda$ we get with probability $1 - \rho$:

$$||w_{i*}^{\lambda} - w_{s}||_{\frac{1}{n}X^{T}X} \le ||\tilde{w}_{s} - w_{s}||_{\Sigma} \le ||\mathcal{X}|||\mathcal{Y}|n||A||_{2} \le ||\mathcal{X}||||Y||\frac{\lambda_{\max}(\Sigma)}{\lambda^{2}}\delta^{2} =$$
(34)

$$2||\mathcal{X}|||\mathcal{Y}|\frac{(\frac{||\mathcal{X}||}{n}+\lambda)^2}{\lambda^3}\frac{\log(2d/\rho)}{s},$$
(35)

which under the assumption $||\mathcal{X}|| = O(n\lambda)$ gives $O(\frac{||\mathcal{X}|||\mathcal{Y}|\log(2d/\rho)}{\lambda s})$. as desired.

Proof of Lemma 13:

Proof. By Lemma 8, with $A = \frac{1}{s}X_{S}^{T}X + \lambda I$, $B = E_{1}, c = E_{2}, v = \frac{1}{n}^{T}Y$, we get $w_{s} - \tilde{w}_{s} = (\frac{1}{s}X_{S}^{T}X + \lambda I + E_{1})E_{1}w_{s} - (\frac{1}{s}X_{S}^{T}X + \lambda I + E_{1})^{-1}E_{2}$, and so

$$||w_s - \tilde{w}_s||_{\frac{X^T X}{n}}^2 \le 2||(\frac{1}{s}X_S^T X + \lambda I + E_1)^{-1}E_1 w_s||_{\frac{X^T X}{n}}^2 + 2||(\frac{1}{s}X_S^T X + \lambda I + E_1)^{-1}E_2||_{\frac{X^T X}{n}}^2$$

Under the assumption $||E_1||_2 \leq \lambda/2$, this becomes

$$||w_{s} - \tilde{w}_{s}||_{\frac{X^{T}X}{n}}^{2} = O\left(||E_{1}w_{s}||_{(\frac{1}{s}X_{S}^{T}X_{S} + \lambda I)^{-1}(X^{T}X/n + \lambda I)(\frac{1}{s}X_{S}^{T}X_{S} + \lambda I)^{-1}}\right) + O\left(||E_{2}||_{(\frac{1}{s}X_{S}^{T}X_{S} + \lambda I)^{-1}(X^{T}X/n + \lambda I)(\frac{1}{s}X_{S}^{T}X_{S} + \lambda I)^{-1}}\right)$$
(36)

Now to apply Lemma 11, we need to bound

$$\operatorname{Tr}\left(\left(\frac{1}{s}X_{S}^{T}X_{S}+\lambda I\right)^{-1}(X^{T}X/n+\lambda I)\left(\frac{1}{s}X_{S}^{T}X_{S}+\lambda I\right)^{-1}\right) \leq d\lambda_{\max}(\Sigma_{S})^{-1}(\Sigma)(\Sigma_{S})^{-1}) = d\lambda_{\max}(\Sigma^{-1/2}(\Sigma^{1/2}\Sigma_{S}^{-1}\Sigma^{1/2})^{2}\Sigma^{-1/2}) \leq d\lambda_{\max}(\Sigma^{-1})\frac{1}{\lambda_{\min}(\Sigma^{-1/2}\Sigma_{S}^{-1}\Sigma^{1/2})}^{2} \leq \frac{d}{\lambda}(\frac{1}{1-\delta})^{2}, \quad (37)$$

where the last inequality follows from Lemma 14. Applying Lemma 11 we get that with probability $1 - 2\rho$:

$$||w_{i*}^{\lambda} - w_{s}||_{X^{T}X/n} = O\left(\sigma_{1}^{2} \cdot \frac{d}{\lambda}(\frac{1}{1-\delta})^{2} \cdot ||w_{s}||^{2}\log(2d^{2}/\rho) + \sigma_{2}^{2} \cdot \frac{d}{\lambda}(\frac{1}{1-\delta})^{2}\log(2d^{2}/\rho)\right), \quad (38)$$

From Lemma 14, $\delta = \sqrt{\frac{2(1+\frac{||\mathcal{X}||}{n\lambda})\log(2d/\rho)}{s}}$, which under the assumption $||\mathcal{X}|| = O(n\lambda)$ gives $\frac{1}{(1-\delta)}^2 = O(1+\frac{\log(2d/\rho)}{s})^2$. Substituting in the value of σ_1, σ_2 gives:

$$\frac{||X||^4n^2}{s^4\epsilon^2}||\mathcal{W}||^2\cdot\frac{d}{\lambda}(1+\frac{\log(2d/\rho)}{s}) + \frac{l||\mathcal{X}||^2||\mathcal{Y}||^2}{n^2\epsilon^2}\cdot\frac{d}{\lambda}(1+\frac{\log(2d/\rho)}{s}),$$

as desired.